参赛队员姓名：张文翰，郑子阳

中学：北京师范大学附属实验中学

省份：北京

国家/地区：中华人民共和国

指导教师姓名：孙晓

论文题目：A Study on Möbius Function and Euler’s Totient of Order K

A STUDY ON MÖBIUS FUNCTION AND EULER'S TOTIENT OF ORDER K

Wenhan Zhang and Ziyang Zheng
The Experimental HS Attached to BNU

ABSTRACT. Defined by Tom M. Apostol (see [2]), the Möbius function of order *k* is a natural generalization of Möbius function which is one of the most important arithmetic functions studied in analytic number theory. In this paper we study some properties of Möbius function of order *k*, denoted by μ_k, some of which are analogous to ordinary Möbius function. This involves some summation formulas involving μ_k. We also use some of them to study "*k*-free integers". And from here, we define Euler's Totient of order *k*, denoted by φ_k, and study its properties and relation with μ_k. We also present asymptotic formula about φ_k with proof. Furthermore, we study the asymptotic behavior of *k*-free integers with the help of μ_k.

KEYWORDS. Möbius function of order *k*; *k*th-power free integers (*k*-free integers); summation and asymptotic formulas; Euler's Totient of order *k*

0. Content

1. Introduction

Möbius function is one very important arithmetic function whose properties are beautiful and pragmatic in connection with the study of other arithmetic functions and analytic number theory. As a natural generalization of Möbius function, the Möbius function of order k, defined originally by T. M. Apostol, is one central topic of this paper.

A key operation between arithmetic functions in number theory is Dirichlet convolution, an operation which is communicative and associative. This paper will not focus on the general Dirichlet convolution, but some properties of the arithmetic functions we discuss is of that exact form or a similar form. And some of such formulas are especially useful to draw connections between different arithmetic functions and deduce asymptotic formulas of partial sum of unfamiliar functions to familiar ones.

Another tool worth mentioning is Riemann zeta function. It is always miraculously connected to all different subfields of analytic number theory. And some of our results make use of $\zeta(s)$'s properties in related to $\mu(n)$ (like $\mu(n)$'s relation with $1/\zeta(s)$). Apart from that, Riemann zeta function appears in some basic asymptotic formulas we make use of as well to deduce more asymptotic formulas, like the partial sum of $\varphi_k(n)$.

We begin with a review of the definition of the ordinary Möbius function $\mu(n)$ in Section 2, and give some elementary properties of it which we will generalize to $\mu_k(n)$. Then we begin our discussion of μ_k in Section 3. After that, we define function φ_k, which we shall call "Euler's Totient of order k" in Section 4. We will also present our results of several formulas relating Möbius function and Euler's Totient of order k.

In Section 5 and 6, we discuss some asymptotic formulas. We begin with its relation with μ_k and a proof of Gegenbauer's theorem of k-free integers (Apostol didn't go into detail of that proof in his paper). With this theorem we easily estimate the "proportion" of k-free integers. And in Section 6 we study asymptotic formulas involving $\varphi_k(n)$.

Note that without specified, the letter p or p_i (p with some subscript) always denote prime numbers in the set of positive integers. And for arithmetic function we mean a mapping from $\mathbb{Z}^+$ to $\mathbb{C}$, as usual. And $\pi = 3.1415926\ldots$

2. A Review of Möbius Function (and Euler's Totient Function)

In this section we present a brief review of some basic known facts about Möbius functions (for details and further see [1]).

Definition 2.1 Möbius function is an arithmetic function $\mu(n)$ defined as

$$\begin{cases} \mu(1) = 1; \\ \mu(n) = (-1)^k \ if\ n\ is\ the\ product\ of\ k\ distinct\ primes; \\ \mu(n) = 0\ otherwise. \end{cases}$$

From the definition we notice that $\mu(n) = 0$ if and only if n is not square free, by which we mean that n contains a divisor which is a square number. A general definition of k-free is the following:

Definition 2.2 An integer n is kth-power free or k-free if there's no prime p such that $p^k \mid n$.

The rest of this section is devoted to the presentation of some properties of $\mu(n)$ without proof. Note that these results are not new; in fact they are special cases of our new results regarding the properties of $\mu_k(n)$ (which we will prove), which we will present in Section 3.

Proposition 2.1

$$\sum_{d|n} \mu(d) = \left\lfloor \frac{1}{n} \right\rfloor$$

Here "$\lfloor x \rfloor$" denotes the greatest integer less than or equal to x, namely the floor function.

Proposition 2.2 *Let $\varphi(n)$ denotes Euler's Totient function (the number of integers coprime to n within $[1, n]$), we have*

$$\varphi(n) = \sum_{d|n} \mu(d) \frac{n}{d}$$

Proposition 2.3 *Let $\varphi(x, n)$ denote the number of positive integers less than or equal to x that are coprime to n. Then*

$$\varphi(x, n) = \sum_{d|n} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor$$

3. Möbius Function of Order *k* and Its Properties

Möbius function of order k is defined by Apostol in the following way:

Definition 3.1 Möbius function of order k is an arithmetic function $\mu_k(n)$ such that:

$$\begin{cases} \mu_k(1) = 1; \\ \mu_k(n) = 0 \text{ } if \text{ } n \text{ } is \text{ } not \text{ ``}(k+1) - free\text{''}; \\ \mu_k(n) = (-1)^r \text{ } if \text{ } n = {p_1}^k \dots {p_r}^k {p_{r+1}}^{a_{r+1}} \dots {p_l}^{a_l}, where \text{ } 0 \le a_{r+1}, \dots, a_l < k; \\ \mu_k(n) = 1 \text{ } otherwise \end{cases}$$

Note that for $k = 1$, $\mu_1(n) = \mu(n)$.

From the properties of Möbius functions and some numerical trials, we have conjectured (and proved, see below) the following formulas regarding $\mu_k(n)$.

Theorem 3.1 *For* $n, k, d \in \mathbb{Z}^+$, *we have:*

$$\sum_{d^k | n} \mu_k(d^k) = \begin{cases} 0 \text{ } if \text{ } n \text{ } is \text{ } not \text{ ``}k - free\text{''}; \\ 1 \text{ } otherwise. \end{cases}$$

Theorem 3.2 *For* $k, d \in \mathbb{Z}^+$, $x \in \mathbb{R}^+$, $x \ge 1$ $\sum_{d \le x} \mu(d) \left\lfloor \frac{x}{d} \right\rfloor = 1$; *and in fact a general case is that*

$$\sum_{d \le x} \mu_k(d^k) \left\lfloor \frac{x}{d^k} \right\rfloor = \#_k(x)$$

Where we denote the number of k-free positive integers less than or equal to x as $\#_k(x)$. *Note that for all positive integer k,* 1 *is k-free. This is from the definition of k-free and the fact that* 1 *is not divisible by any positive power prime numbers. So the first formula is the special case of the second one* (k = 1 *case*).

Apostol studied several properties of $\mu_k(n)$ himself, the following formula is one of them which turns out to be a very important tool in our study.

Lemma 3.1 *For all* $k \ge 1$ *and* $n \in \mathbb{Z}^+$, $\mu_k(n^k) = \mu(n)$. *Here* $\mu(n)$ *is the Möbius function* (*of order* 1).

PROOF. By definition, $\mu_k(1^k) = \mu_k(1) = 1 = \mu(1)$; if n is square-free, let $n = p_1 \dots p_r$, so $n^k = {p_1}^k \dots {p_r}^k$ and therefore $\mu_k(n^k) = (-1)^r = \mu(n)$. Now if n is not square-free, there exist prime p such that $p^2|n$, so $p^{2k}|n^k$. For all $k \geq 1, 2k \geq k+1$, so $p^{k+1}|n^k$. And by definition, $\mu_k(n^k) = 0 = \mu(n)$ in this case. \\\

We will now prove our Theorem 3.1 and 3.2. Note that with the above lemma μ_k's in our conjectures can be reduced to our familiar μ.

PROOF OF **Theorem 3.1**. By Lemma 3.1, it is equivalent to prove that for $n, k, d \in \mathbb{Z}^+$,

$$\sum_{d^k|n} \mu(d) = \begin{cases} 0 \; if \; n \; is \; not \; \text{“}k - free\text{”}; \\ 1 \; otherwise. \end{cases}$$

If n is not k-free, write $n = {p_1}^{a_1} \dots {p_r}^{a_r} {p_{r+1}}^{a_{r+1}} \dots {p_l}^{a_l}$ where $a_1, \dots, a_r \geq k$, and $0 \leq a_{r+1}, \dots, a_l \leq k$. So all the none zero terms *LHS* are:

$$\sum_{d^k|n} \mu(d) = \mu(1) + \mu(p_1) + \cdots + \mu(p_r) + \mu(p_1 p_2) + \cdots + \mu(p_1 \dots p_r)$$

$$= \binom{r}{0}(-1)^0 + \binom{r}{1}(-1)^1 + \cdots + \binom{r}{i}(-1)^i + \binom{r}{r}(-1)^r$$

$$= [(x-1)^r]_{x=1} = 0.$$

And if n is k-free, then the only d such that $d^k|n$ is 1. So

$$\sum_{d^k|n} \mu(d) = \mu(1) = 1.$$

\\\

PROOF OF **Theorem 3.2**. By Lemma 3.1, it is equivalent to prove that for $k, d \in \mathbb{Z}^+$, $x \in \mathbb{R}^+$,

$$\sum_{d \leq x} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor = \#_k(x)$$

We have:

$$\sum_{d \leq x} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor = \sum_{d \leq x} \sum_{\substack{d^k|y \\ 1 \leq y \leq x}} \mu(d) = \sum_{1 \leq y \leq x} \sum_{d^k|y} \mu(d)$$

where the last step is obtained by changing the order of summation. Note that by Theorem 3.1, $\sum_{d^k|y} \mu(d)$ is 1 if and only if y is k-free, and is 0 otherwise. So $\sum_{1\le y\le x} \sum_{d^k|y} \mu(d)$ counts all k-free positive integers less than or equal to x. \\\

Apostol proved (see [1]) that each $\mu_k(n)$ is multiplicative. We can now consider another summation and make use of this result:

$$\sum_{d|n} \mu_k(d)$$

Let $n = {p_1}^{a_1} \dots {p_r}^{a_r}$, each $a_i > 0$, let $d = {p_1}^{b_1} \dots {p_r}^{b_r}$, each $b_i \ge 0$. So

$$\mu_k(d) = \mu_k({p_1}^{b_1}) \dots \mu_k({p_r}^{b_r})$$

So

$$\sum_{d|n} \mu_k(d) = \sum_{d|{p_1}^{a_1}} \mu_k(d) \dots \sum_{d|{p_r}^{a_r}} \mu_k(d)$$

It is much easier to compute each summation $\sum_{d|{p_i}^{a_i}} \mu_k(d)$, this is

$$\sum_{d|{p_i}^{a_i}} \mu_k(d) = \begin{cases} a_i + 1 \ if \ a_i < k; \\ k - 1 \ if \ a_i \ge k. \end{cases}$$

Also we can study:

$$h_k(n) = \sum_{d|n} \mu_k(d) \frac{n}{d}$$

by first proving that $h_k(n)$ is a multiplicative function for each n. Let $n = {p_1}^{a_1} \dots {p_r}^{a_r}$ and $d = {p_1}^{b_1} \dots {p_r}^{b_r}$, where $a_i > 0$ and $b_i \ge 0$. We have

$$\sum_{d|n} \mu_k(d) \frac{n}{d} = \sum_{b_1=0}^{a_1} \dots \sum_{b_r=0}^{a_r} \left(\mu_k({p_1}^{b_1}) \frac{{p_1}^{a_1}}{{p_1}^{b_1}}\right) \dots \left(\mu_k({p_r}^{b_r}) \frac{{p_r}^{a_r}}{{p_r}^{b_r}}\right)$$

$$= \prod_{t=1}^{r} \left(\sum_{b_t=0}^{a_t} \mu_k({p_t}^{b_t}) \frac{{p_t}^{a_t}}{{p_t}^{b_t}} \right) = \prod_{t=1}^{r} h_k({p_t}^{a_t})$$

With this relation the multiplicity simply follows. So we can simply study

$$h_k({p_t}^{a_t}) = \sum_{d|{p_t}^{a_t}} \mu_k(d) \frac{{p_t}^{a_t}}{d}$$

This is much easier to calculate:

$$h_k({p_t}^{a_t}) = \sum_{d|{p_t}^{a_t}} \mu_k(d)\frac{{p_t}^{a_t}}{d} = \left(\sum_{i=0}^{k-1} \lfloor {p_t}^{a_t-i} \rfloor\right) - \lfloor {p_t}^{a_t-k} \rfloor$$

which follows from the definition and a similar argument as the above discussion.

4. Euler's Totient of Order *k* and Its Properties

Euler's Totient of order k is our generalization of Euler's Totient. We will present our definition of Euler's Totient of order k first, and then discuss a bit about how it comes.

Definition 4.1 For a positive integer k, we define Euler's Totient of order k, denoted by φ_k, as an arithmetic function that $\varphi_k(n)$ = the number of positive integers i less than or equal to n such that gcd(n, i) is k-free. Here "gcd(n, i)" denotes the greatest common divisor of n and i.

The original idea of such definition comes from Proposition 2.2, and we want to generalize the Dirichlet convolution to the Möbius function of order k. But what would such generalization mean, as an arithmetic function of n? We have conjectured and proved that it is the above Euler's Totient of order k. This is illustrated in our theorem below:

Theorem 4.1 *For* $n, k, d \in \mathbb{Z}^+$, *we have:*

$$\varphi_k(n) = \sum_{d^k|n} \mu_k(d^k)\frac{n}{d^k} = \sum_{d^k|n} \mu(d)\frac{n}{d^k}$$

Note that the second equality follows from Lemma 3.1.

Actually this result is a consequence of our next theorem (Theorem 4.2), which is our generalization of Proposition 2.3. We first generalize $\varphi(x, n)$ into $\varphi_k(x, n)$:

Definition 4.2 For $x \in \mathbb{R}^+$ and $n \in \mathbb{Z}^+$, define $\varphi_k(x, n)$ = the number of positive integers i less than or equal to x such that gcd(n, i) is k-free.

Theorem 4.2 *For* $n, k, d \in \mathbb{Z}^+$ *and* $x \in \mathbb{R}^+$ *we have:*

$$\varphi_k(x,n) = \sum_{d^k|n} \mu_k(d^k)\left\lfloor\frac{x}{d^k}\right\rfloor = \sum_{d^k|n} \mu(d)\left\lfloor\frac{x}{d^k}\right\rfloor$$

Note that Theorem 4.1 *is a special case of the theorem above, by setting x = n.*

PROOF. Clearly, $\left\lfloor\frac{x}{d^k}\right\rfloor$ counts the number of positive integers less than or equal to *x* that are the multiple of d^k. Let $d^k = ({p_1}^k)^{a_1} \dots ({p_r}^k)^{a_r}$, where $p_1, \dots, p_r$ are distinct prime factors of *n* whose *k*th power divides *n* and $a_i \geq 0$. Since $d^k = ({p_1}^{a_1} \dots {p_r}^{a_r})^k$ Hence $\sum_{d^k|n} \mu(d)\left\lfloor\frac{x}{d^k}\right\rfloor$ counts:

$$\begin{aligned} &all\ integers\ within\ [1,x] \\ &\qquad -\sum number\ of\ multiples\ of\ each\ {p_i}^k \\ &\qquad +\sum number\ of\ multiples\ of\ each\ (p_i p_j)^k \\ &\qquad -\sum number\ of\ multiples\ of\ each\ (p_i p_j p_l)^k + \cdots \\ &\qquad + (-1)^r \times number\ of\ multiples\ of\ (p_1 \dots p_r)^k \end{aligned}$$

by the definition of Möbius function (the other terms which involves $a_i > 1$ all vanishes). Hence by Inclusion-Exclusion Principle, $\sum_{d^k|n} \mu(d)\left\lfloor\frac{x}{d^k}\right\rfloor$ counts all integers within $[1,x]$ that are not multiple of any p_i, each counted without repetition. And that is the number of integers in that interval whose greatest common divisor with *n* is *k*-free, so $\sum_{d^k|n} \mu(d)\left\lfloor\frac{x}{d^k}\right\rfloor = \varphi_k(x,n)$. \\\

5. Asymptotic Formulas Related to *k*-free Integers and Möbius functions

We begin our discussion of asymptotic formulas from this Section on, which requires some tools in analytic number theory. We shall first give some known results as lemmas which we shall make frequent use of (for details and further see [1]).

Lemma 5.1 *For* $s > 1$, $\frac{1}{\zeta(s)} = \sum_{n\geq 1} \frac{\mu(n)}{n^s}$, *where* $\mu(n)$ *is the Möbius function,* $n \in \mathbb{Z}^+$.

Note that we shall only make use of the real-variable Riemann zeta function $\zeta(s)$, defined as $\zeta(s) = \sum_{n\geq 1}\frac{1}{n^s}$, $n \in \mathbb{Z}^+$. And as commonly known it converges when $s > 1$.

Lemma 5.2 *Let* $x \geq 1$ *and* $n \in \mathbb{Z}^+$, *we have:*

$$\sum_{n\leq x}\frac{1}{n^s} = \begin{cases} \log x + C + O\left(\frac{1}{x}\right), if\ s = 1; \\ \frac{x^{1-s}}{1-s} + \zeta(s) + O(x^{-s}), if\ s > 0\ and\ s \neq 1. \end{cases}$$

Here we shall explain the notations. "log *x*" always denote natural logarithm (with base *e*). *C* here denotes the Euler's Constant, and $C = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} - \log n\right)$. The "big-oh" notation is defined such that $f(x) = O(g(x))$ where $g(x)$ is positive-valued means that for sufficiently large *x*, there exist a constant $M > 0$ such that $|f(x)| \leq Mg(x)$. And an equation like $f(x) = g(x) + O\big(h(x)\big)$ means that $f(x) - g(x) = O(h(x))$.

Lemma 5.3 *Let* $x \geq 1$ *and* $n \in \mathbb{Z}^+$, *we have:*

$$\sum_{n>x}\frac{1}{n^s} = O(x^{1-s})$$

for $s > 1$.

Recall that in Theorem 3.2 we find that number of *k*-free positive integers less than or equal to *x* (a real number greater than or equal to 1) is $\sum_{d\leq x}\mu(d)\left\lfloor\frac{x}{d^k}\right\rfloor$. Denote the number of *k*-free positive integers less than or equal to *x* as $\#_k(x)$, so

$$\#_k(x) = \sum_{d\leq x}\mu(d)\left\lfloor\frac{x}{d^k}\right\rfloor.$$

We will first try to find an asymptotic formula based on this, and see whether it is helpful. Note that we assumed *k* to be greater than 1, or else we have $\#_k(x) = 1$ for *k* = 1.

$$\#_k(x) = \sum_{d\le x} \mu(d) \left\lfloor \frac{x}{d^k} \right\rfloor = \sum_{d\le x} \mu(d) \left(\frac{x}{d^k} + O(1) \right) = \left(\sum_{d\le x} \mu(d) \frac{x}{d^k} \right) + O(x)$$

$$= x \left(\sum_{d\ge 1} \frac{\mu(d)}{d^k} - \sum_{d>x} \frac{\mu(d)}{d^k} \right) + O(x)$$

$$= x \left(\sum_{d\ge 1} \frac{\mu(d)}{d^k} + O\left(\sum_{d>x} \frac{1}{d^k} \right) \right) + O(x)$$

$$= x \left(\frac{1}{\zeta(k)} + O(x^{1-k}) \right) + O(x) = \frac{x}{\zeta(k)} + O(x) = O(x)$$

Unfortunately, this result is not so helpful, due to the relatively large error bound of $O(x)$. But we can do it in another way.

Lemma 5.4 *For* $n, k, d \in \mathbb{Z}^+$*, we have:*

$$|\mu_k(n)| = \sum_{d^{k+1}|n} \mu(d)$$

PROOF. We prove by several case works on *n*. If *n* = 1, Clearly

$$|\mu_k(1)| = 1 = \mu(1) = \sum_{d^{k+1}|1} \mu(d)$$

If $p_1, \ldots, p_l$ are all the primes such that ${p_i}^{k+1}|n$, $|\mu_k(n)| = 0$. And

$$\sum_{d^{k+1}|n} \mu(d) = \mu(1) + \mu(p_1) + \cdots + \mu(p_1 p_2) + \cdots + \mu(p_1 \ldots p_l) = 0$$

As shown before in the proof of Theorem 3.1. Now if $n = {p_1}^k \ldots {p_r}^k {p_{r+1}}^{a_{r+1}} \ldots {p_l}^{a_l}$, where $0 \le a_{r+1}, \ldots, a_l < k$, then $|\mu_k(n)| = 1$, and $\sum_{d^{k+1}|1} \mu(d) = \mu(1) = 1$. And if $n = {p_1}^{a_1} \ldots {p_r}^{a_r}$ where $0 \le a_1, \ldots, a_r < k$, then $|\mu_k(n)| = 1$ and $\sum_{d^{k+1}|1} \mu(d) = \mu(1) = 1$ as well. And our proof is complete. \\\

Our proof is a different one with Apostol's proof in his paper. So we go into some more details above.

Now by Theorem 3.1, we know that $\sum_{d^{k+1}|n} \mu(d) = 1$ if and only if *n* is (*k* + 1)-free and equals to 0 otherwise. So we have (*k* is positive integer)

$$\#_{k+1}(x) = \sum_{n\le x} \sum_{d^{k+1}|n} \mu(d) = \sum_{n\le x} |\mu_k(n)|$$

This is the formula that related (k + 1)-free positive integers and function $\mu_k(n)$. And we can further derive Gegenbauer's theorem of k-free integers with

$$\#_{k+1}(x) = \sum_{n\le x}\sum_{d^{k+1}|n}\mu(d) = \sum_{\substack{q,d^{k+1}\\ qd^{k+1}\le x}}\mu(d) = \sum_{d^{k+1}\le x}\left(\mu(d)\sum_{q\le\frac{x}{d^{k+1}}}1\right)$$

$$= \sum_{d^{k+1}\le x}\left(\mu(d)\left\lfloor\frac{x}{d^{k+1}}\right\rfloor\right) = \sum_{d^{k+1}\le x}\left(\mu(d)(\frac{x}{d^{k+1}}+O(1)\right)$$

$$= \sum_{d\le x^{\frac{1}{k+1}}}\mu(d)\frac{x}{d^{k+1}} + \sum_{d\le x^{\frac{1}{k+1}}}\left(\mu(d)+O(1)\right)$$

$$= \sum_{d\le x^{\frac{1}{k+1}}}\mu(d)\frac{x}{d^{k+1}} + O\left(x^{\frac{1}{k+1}}\right) = x\sum_{d\le x^{\frac{1}{k+1}}}\frac{\mu(d)}{d^{k+1}} + O\left(x^{\frac{1}{k+1}}\right)$$

$$= x\left(\sum_{d=1}^{\infty}\frac{\mu(d)}{d^{k+1}} - \sum_{d>x^{\frac{1}{k+1}}}\frac{\mu(d)}{d^{k+1}}\right) + O\left(x^{\frac{1}{k+1}}\right)$$

$$= x\left(\frac{1}{\zeta(k+1)} + O\left(\sum_{d>x^{\frac{1}{k+1}}}\frac{1}{d^{k+1}}\right)\right) + O\left(x^{\frac{1}{k+1}}\right)$$

$$= x\left(\frac{1}{\zeta(k+1)} + O\left(x^{\frac{-k}{k+1}}\right)\right) + O\left(x^{\frac{1}{k+1}}\right) = \frac{x}{\zeta(k+1)} + O\left(x^{\frac{1}{k+1}}\right)$$

which is exactly what Gegenbauer's theorem states. So now we have

$$\#_{k+1}(x) = \sum_{n\le x}|\mu_k(n)| = \frac{x}{\zeta(k+1)} + O\left(x^{\frac{1}{k+1}}\right)$$

Therefore we can easily deduce the "proportion" of (k + 1)-free integers (k is positive integer):

$$\lim_{x\to\infty}\frac{\#_{k+1}(x)}{x} = \frac{\frac{x}{\zeta(k+1)} + O\left(x^{\frac{1}{k+1}}\right)}{x} = \frac{1}{\zeta(k+1)}.$$

Loosely speaking, that is to say if we choose a positive integer randomly, the probability that this integer is (k + 1)-free is $\frac{1}{\zeta(k+1)}$. Or the "proportion" of (k + 1)-free integers among all positive integers is $\frac{1}{\zeta(k+1)}$. A special case is k = 1, which shows that the "proportion" of square-free integer among all positive integers is $\frac{1}{\zeta(2)}$, which is $\frac{6}{\pi^2}$. This is approximately 0.60793, so a slightly more that 60% of positive integers are square-free, which is more than a half.

6. Asymptotic Formula Involving φ_k

In this Section the goal is to study the average order of φ_k. What we will do is to deduce an asymptotic formula for its partial sum, namely $\sum_{n\le x}\varphi_k(n)$. Here k and n are positive integers and x is a real number greater than 1.

We begin with relation

$$\varphi_k(n)=\sum_{d^k|n}\mu_k(d^k)\frac{n}{d^k}=\sum_{d^k|n}\mu(d)\frac{n}{d^k}$$

which our Theorem 4.1. So we have

$$\sum_{n\le x}\varphi_k(n)=\sum_{n\le x}\sum_{d^k|n}\mu(d)\frac{n}{d^k}$$

Therefore,

$$\sum_{n\le x}\varphi_k(n)=\sum_{n\le x}\sum_{d^k|n}\mu(d)\frac{n}{d^k}=\sum_{\substack{q,d^k\\ qd^k\le x}}\mu(d)q=\sum_{d^k\le x}\left(\mu(d)\sum_{q\le\frac{x}{d^k}}q\right)$$

$$=\sum_{d\le x^{\frac{1}{k}}}\left(\mu(d)\left(\frac{1}{2}\left(\frac{x}{d^k}\right)^2+O\left(\frac{x}{d^k}\right)\right)\right)$$

Here we use the simple fact that $\sum_{q\le\frac{x}{d^k}}q=\frac{1}{2}\left(\frac{x}{d^k}\right)^2+O\left(\frac{x}{d^k}\right)$. This is because

$$\sum_{q\le\frac{x}{d^k}}q=\frac{\left\lfloor\frac{x}{d^k}\right\rfloor\left(\left\lfloor\frac{x}{d^k}\right\rfloor+1\right)}{2}=\frac{\left(\frac{x}{d^k}+O(1)\right)^2}{2}=\frac{\left(\frac{x}{d^k}\right)^2+O\left(\frac{x}{d^k}\right)+O(1)}{2}$$

$$=\frac{1}{2}\left(\frac{x}{d^k}\right)^2+O\left(\frac{x}{d^k}\right)$$

as wanted. So up to now we have

$$\sum_{n\le x}\varphi_k(n)=\frac{1}{2}x^2\left(\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^{2k}}\right)+O\left(x\sum_{d\le x^{\frac{1}{k}}}\frac{1}{d^k}\right)$$

Now there are two parts that need to be reduced. One is $\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^{2k}}$, and the other one is $x\sum_{d\le x^{\frac{1}{k}}}\frac{1}{d^k}$ inside the big-oh notation. We first figure out the "$\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^{2k}}$" term.

We write

$$\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^{2k}}=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^{2k}}-\sum_{d>x^{\frac{1}{k}}}\frac{\mu(d)}{d^{2k}}$$

where the first term is just $\frac{1}{\zeta(2k)}$ by Lemma 5.1. And we continue to estimate the second term:

$$\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^{2k}}=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^{2k}}-\sum_{d>x^{\frac{1}{k}}}\frac{\mu(d)}{d^{2k}}=\sum_{d=1}^{\infty}\frac{\mu(d)}{d^{2k}}+O\left(\sum_{d>x^{\frac{1}{k}}}\frac{1}{d^{2k}}\right)$$

$$=\frac{1}{\zeta(2k)}+O\left(\left(x^{\frac{1}{k}}\right)^{1-2k}\right)$$

The last step follows from Lemma 5.3, since we know that $2k>1$ for positive integer k. So we have:

$$\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^{2k}}=\frac{1}{\zeta(2k)}+O\left(x^{\frac{1-2k}{k}}\right)$$

This is an asymptotic formula related to Möbius function, which is our generalization of the known one:

$$\sum_{d\le x}\frac{\mu(d)}{d^2}=\frac{1}{\zeta(2)}+O\left(\frac{1}{x}\right)=\frac{6}{\pi^2}+O\left(\frac{1}{x}\right)$$

We now turn to our discussion of $x\sum_{d\le x^{\frac{1}{k}}}\frac{1}{d^k}$. We here make use of Lemma 5.2. So this requires some case work: the $k=1\ (so\ \frac{1}{k}=1)$ case and the $k>1\ (\frac{1}{k}\neq 1\ and\ \frac{1}{k}>0)$ case. For $k = 1$ we will get the known asymptotic formula of $\sum_{n\le x}\varphi(n)$. Here we shall first repeat that proof:

$$\sum_{n\le x}\varphi(n)=\frac{1}{2}x^2\left(\sum_{d\le x}\frac{\mu(d)}{d^2}\right)+O\left(x\sum_{d\le x}\frac{1}{d}\right)$$

$$=\frac{1}{2}x^2\left(\frac{6}{\pi^2}+O\left(\frac{1}{x}\right)\right)+O\left(x\left(\log x+C+O\left(\frac{1}{x}\right)\right)\right)$$

$$=\frac{3}{\pi^2}x^2+O(x\log x)$$

And for $k>1$, we present our generalization here. We make use of the other formula in Lemma 5.2, so we have

$$\sum_{n\le x}\varphi_k(n)=\frac{1}{2}x^2\left(\sum_{d\le x^{\frac{1}{k}}}\frac{\mu(d)}{d^{2k}}\right)+O\left(x\sum_{d\le x^{\frac{1}{k}}}\frac{1}{d^k}\right)$$

$$=\frac{1}{2}x^2\left(\frac{1}{\zeta(2k)}+O\left(x^{\frac{1-2k}{k}}\right)\right)+O\left(x\left(\frac{x^{\frac{1-k}{k}}}{1-k}+\zeta(k)+O(x^{-1})\right)\right)$$

$$=\frac{x^2}{2\zeta(2k)}+O\left(x^{\frac{1}{k}}\right)+O(x)+O(1)=\frac{x^2}{2\zeta(2k)}+O(x)$$

Note that our error term for $k>1$ ($O(x)$) is actually smaller than the error term of $k=1$ known case. So it is a relatively good estimation with acceptable error bound.

Putting the results together, we have the following theorem:

Theorem 6.1 *For* $k,n\in\mathbb{Z}^+, x\in\mathbb{R}^+$ *and* $x\ge 1$, *we have:*

$$\sum_{n\le x}\varphi_k(n)=\begin{cases}\frac{3}{\pi^2}x^2+O(x\,log\,x)\ if\ k=1;\\ \frac{x^2}{2\zeta(2k)}+O(x)\ if\ k>1.\end{cases}$$

where the $k=1$ *case is a known result and the* $k>1$ *is our new generalization of the Euler's Totient of order k.*

7. Some Other Discussions

Another question we can ask is, given any *n* positive integers, what is the "probability" that their greatest common divisor is *k*-free? Note that it is a generalization of the problem which asks the "probability" of *n* given integers being coprime.

Theorem 7.1 *Given any n positive integers, the "probability" that their greatest common divisor is k-free, equals to* $\frac{1}{\zeta(nk)}$.

PROOF. The greatest common divisor of n positive integers being k-free means that for any prime p, none of them is a multiple of p^k. The "probability" that an integer IS a multiple of p^k is $\frac{1}{p^k}$, so the "probability" that n integers is NOT all multiple of p^k is $1 - \frac{1}{p^{nk}}$. And for different prime powers such "probabilities" are independent. Therefore the probability that n integers' gcd is k-free is

$$\prod_{p\ prime}\left(1-\frac{1}{p^{nk}}\right)=\frac{1}{\frac{1}{\prod_{p\ prime}\left(1-\frac{1}{p^{nk}}\right)}}$$

Since $\zeta(s)=\frac{1}{\prod_{p\ prime}\left(1-\frac{1}{p^s}\right)}$, therefore our desired "probability" is $\frac{1}{\zeta(nk)}$. \\\

At last, we want to talk briefly about some further related topics. Our source of inspiration, of Euler's totient of order k, is the Dirichlet convolution connecting Möbius function and Euler's Totient:

$$\varphi(n)=\sum_{d|n}\mu(d)\frac{n}{d}$$

But this is not the only such formula involving $\mu(d)$. The point is for each such formula, there's a potential to define a new arithmetic function of order k, and study its properties for fun or for further estimations in analytic number theory. For example, we may define the Mangoldt function of order k. Recall the definition of Mangoldt function $\Lambda(n)$:

$$\Lambda(n)=\begin{cases}\log p, if\ n=p^m, m\geq 1;\\ 0, otherwise.\end{cases}$$

And a convolution related to Möbius function:

$$\Lambda(n)=\sum_{d|n}\mu(d)\log\frac{n}{d}$$

We may define Mangoldt's function of order k by:

$$\Lambda_k(n) = \sum_{d^k|n} \mu(d) \log \frac{n}{d^k}$$

(or by some other ways?) But whether such functions are useful remains to be discovered, and that's not the central topic of this paper.

We may also note that, there is a product formula for $\varphi_k(n)$, which is a generalization to the one of $\varphi(n)$:

Theorem 7.2 *For* $n, k \in \mathbb{Z}^+$*, we have:*

$$\varphi_k(n) = n \prod_{p^k|n} \left(1 - \frac{1}{p^k}\right)$$

PROOF. Let $n = \prod_{j=1}^{r} p_j{}^{a_j}$ where $a_j > 0$. Expand the *RHS* yields:

$$n \prod_{p^k|n} \left(1 - \frac{1}{p^k}\right) = n \left(1 - \prod_{p^k|n} \frac{1}{p^k} + \prod_{\substack{p_i{}^k|n \\ p_j{}^k|n}} \frac{1}{p_i{}^k p_j{}^k} - \cdots + (-1)^r \prod_{all\ p_j{}^k|n} \frac{1}{\left(\prod_{j=1}^{r} p_j\right)^k}\right)$$

$$= n \left(\mu(1) + \prod_{p^k|n} \frac{\mu(p)}{p^k} + \prod_{\substack{p_i{}^k|n \\ p_j{}^k|n}} \frac{\mu(p_i p_j)}{p_i{}^k p_j{}^k} + \cdots + \prod_{all\ p_j{}^k|n} \frac{\mu\left(\prod_{j=1}^{r} p_j\right)}{\left(\prod_{j=1}^{r} p_j\right)^k}\right)$$

$$= n \sum_{d^k|n} \frac{\mu(d)}{d^k} = \sum_{d^k|n} \mu(d) \frac{n}{d^k} = \varphi_k(n)$$

by Theorem 4.1. \\\

8. Sum-up and Ending Note

In this entire paper, we have gotten the following results. First, we study some convolution-type formulas of the Möbius function of order k (Theorem 3.1 and 3.2). Both of them can be used to define "counting functions" of k-free integers smaller than or equal to a given positive real number $x > 1$. Next, we define "Euler's Totient of order k", and give two convolution-type formulas regarding its connection with Möbius function (Theorem 4.1 and 4.2).

Then we turn to the study of analytic properties. With the help of Theorem 3,1 and 3.2, we explore the asymptotic formulas of function $\#_k(x)$, which counts the k-free positive integers less than x and find a proof of Gegenbauer's theorem of k-free integers. After that we go on to discuss $\varphi_k(n)$, and discover the asymptotic formula of its partial sum (Theorem 6.1). Lastly, we make some further discussions to related topics, like some probabilistic problem and product formula of $\varphi_k(n)$.

Throughout the paper, Möbius function of order k, though not explicitly shows up everywhere, is our basic source of inspiration and a thread connecting all of our discussions. The importance of our study is that our results complement the studies of regarding Möbius function of order k by connecting it with other arithmetic functions. We define $\varphi_k(n)$ and gives a thorough studies of its properties and asymptotic formulas of its partial sums. I hope our results can be not merely a study of some functions in analytic number theory, but be a typical example of generalizing arithmetic functions to a broader background as well, just like what we do to Euler's Totient.

As a note for future studies: if we keeps going on with the idea of order k generalization, perhaps we could discover a generalized result for Prime Number Theorem (of "order k primes", maybe). Perhaps this can start from a generalization of Mangoldt function, and hence Chebyshev's functions, and some equivalence statements of PNT. For the studies in analytic number theory, the sky's the limit!

References:

[1] Tom M. Apostol. *Introduction to Analytic Number Theory*. Undergraduate Texts in Mathematics, Photocopy edition. Beijing World Publishing Corporation, Beijing.

[2] Tom M. Apostol. "Möbius Functions of Order k". Pacific Journal of Mathematics, Vol. 32, No. 1, 1970. Accessed on website *projecteuclid.org/download/pdf_1/euclid.pjm/1102977519*.

Acknowledgments

The original inspiration of our studies came from Apostol's idea stated in [2], and some in [1]. What we intended to do is to provide a thorough understanding of Möbius function of order k, as a powerful tool in attacking problems related to the ordinary Möbius function, convolution-type formulas, and asymptotic formulas in analytic number theory. And the original idea of Euler's Totient of order k came from the author Wenhan Zhang's idea in generalizing the Dirichlet convolution connecting Möbius function and Euler's Totient (part of Theorem 4.1), while author Ziyang Zheng proposes Definition 4.1, namely defines the Euler's Totient of order k. Both authors contribute the ideas and proofs in Section 3 and 4, and Ziyang Zheng ran numerical checks on computer to make or check conjectures.

The first author of this paper, Wenhan Zhang is a participant in Ross/Asia Mathematics Program 2019, and wants to give the sincerest acknowledgment to all the Professors, counselors, junior counselors and peers in Ross. Although they didn't help with this paper itself, the program and all of those people inspired the author and helped the author to make remarkable progress in number theory and other areas of mathematics. And apart from some contributions mentioned above, Wenhan Zhang also discovered and proved the asymptotic formulas in Section 5 and 6. And Ziyang Zheng's help in correcting a few mistakes in the first draft of Wenhan's proof and giving some advices in ameliorating or clarifying the argument are also extremely valuable.

We shall also, give our sincerest acknowledgment to our school math teacher Ms. Sun Xiao, as our instructor of this research. She encouraged us a lot in our process of discovering, and gave valuable advices when we were trying to settle down the topic as well as some general directions of several Sections. (All of her helps are non-profit.)

Wenhan Zhang and Ziyang Zheng

September, 2019

此页为学术诚信声明

本参赛团队声明所提交的论文是在指导老师指导下进行的研究工作和取得的研究成果。尽本团队所知，除了文中特别加以标注和致谢中所罗列的内容以外，论文中不包含其他人已经发表或撰写过的研究成果。若有不实之处，本人愿意承担一切相关责任。

参赛队员：张文翰 郑子阳　　指导老师：孙小晓

2019年9月14日