Properties to Determine Inscribed Ellipses of Polygons

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Abstract

In this paper, we extend the result of [1] by calculating some examples in detail, including the inscribed ellipses in triangles, quadrilaterals, and pentagons. We also improve the original proof and reduce the requirements through projective geometry methods in the quadrilateral and pentagon cases. Furthermore, we see the inscribed ellipse problems from the perspective of two projective planes simultaneously, which offers a new way to determine the inscribed ellipses in triangles. Also, we use python to realize the method provided in this paper of drawing inscribed ellipse.

Keywords: Projective Geometry; Inscribed Ellipse.

Contents

1 Introduction

The ellipses is an important component of high school geometry and we often use Geogebra, a drawing tool, to visualize ellipses. However, when we try to draw an inscribed ellipse of a polygon, we can not draw it directly by assigning the tangent points. We can only draw the ellipse first by assigning five distinct points and then draw the subscribe polygon. We can not assign the polygon first. This paper aim to investigate the problem of inscribed ellipse from the view of tangent line by using projective geometry methods [3] [4] [5] and technics [2].

1.1 Homogenous Coordinate of Points and Lines in Projective Plane

Projective Plane is the extension of Euclidean Plane. If we add infinite points and infinite line to the Euclidean Plane, we will get a Projective Plane. Each group of parallel lines in the Projective Plane is defined to meet at a unique infinite point. All the infinite points will compose the infinite line.

Let R be the field of real number and R^2 will stands for Euclidean plane. Let P^2 be the real projective plane. For each point (x, y) in R^2, we associate it with its homogenous point $[x : y : 1]$ in the P^2. For each line $ax+by+c = 0$ in R^2, we associate it with its homogeneous line $ax+by+ch = 0$ in P^2. On line $ax + by + ch = 0$ lies point $[-b : a : 0]$, which is the infinite point of this line. All the infinite points lie on line $h = 0$ at infinity. Since $\forall k \in Z$ and $k \neq 0, kax + kby + kch = 0$ represents the same line as $ax+by+ch = 0$. We can represent a line using its homogenous coordinate $[a : b : 1]$.

1.2 Duality

We can set up a unique dual relationship between the point Q on xyh-plane and the line L_Q on $\alpha\beta\gamma$-plane.

$$Q = [x : y : h] \Longleftrightarrow L_Q = Q \cdot (\alpha, \beta, \gamma)$$

Similarly, there is a unique dual relationship between the line L_P on xyh-plane and the point P on $\alpha\beta\gamma$-plane.

$$L_P = P \cdot (x, y, h) \Longleftrightarrow P = [\alpha : \beta : \gamma]$$

Notice that we can get the coordinate of a point by finding the gradient of the line: $\nabla L_P = P$ and $\nabla L_Q = Q$.

Therefore, for a homogenous curve $\varphi(\alpha, \beta, \gamma)$, we can define its homogenous dual curve: $\hat{\varphi}(x, y, z)$ as

$$\hat{\varphi} = \{[x : y : z] \mid \exists(\alpha, \beta, \gamma) \in \varphi, \text{ such that } (x, y, z) \cdot (\alpha, \beta, \gamma) = 0\}$$

So the homogenous coordinate of $\hat{\varphi}$ is same as $\nabla\varphi$. Therefore, we can get $\hat{\varphi}$ by calculating $\nabla\varphi$.

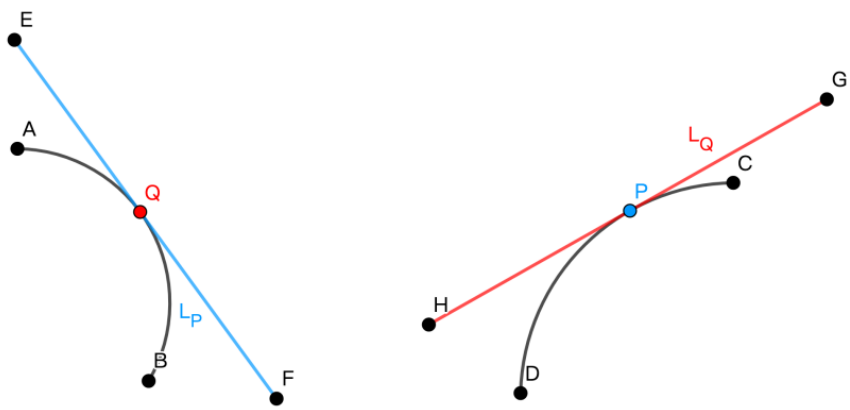

Figure 1: Explanation of Duality

1.3 Conic Curve

Since in this paper we mainly focus on conic curves, this section will introduce some basic knowledge of a conic curve.

Definition 1 *The collections of points* (x_1, x_2, x_3) *that satisfy* $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2a_{12}x_1x_2 + 2a_{13}x_1x_3 + 2a_{23}x_2x_3 = 0$ *are known as conic curves. Here* $a_{ij}(1 \leq i < j \leq 3)$ *are real numbers.*

The conic curves can also be represented as

$$F(x_1, x_2, x_3) = (x_1, x_2, x_3) \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

We often write $\begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{pmatrix}$ as A.

1.3.1 Tangent Line of Conic Curve

Let point $P(p_1, p_2, p_3)$ be a point on conic curve

$$S : (x_1, x_2, x_3)A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

Then the equation for the tangent line at P is

$$(p_1, p_2, p_3)A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = 0$$

2 Duality and Inscribed Ellipses

2.1 Notations and Linfield's Function

We will use the homogenous coordinate of points and lines defined in Chapter 1 for later calculation. Here we define homogenous curve $\varphi = \varphi(\alpha, \beta, \gamma) \in P^2$ and its homogenous dual curve $\hat{\varphi} = \hat{\varphi}(x, y, h) \in P^2$.

We want to find the inscribed ellipse in polygon $Q_1Q_2Q_3 \cdots Q_n$. The main idea of the method is to find the ellipse φ that pass the dual points of the sides of the polygon $Q_1Q_2Q_3 \cdots Q_n : P_{ij}$, where $1 \leq i < j \leq n$. Then, according to definition of dual curve, the dual curve $\hat{\varphi}$ must be tangent to the sides of the polygon.

In this paper, we will use Linfield's function [1] as a way to represent φ :

$$\varphi = \sum_{i=1}^{n} m_i L_1 L_2 \cdots L_{i-1} L_{i+1} \cdots L_n$$

Here L_i is the dual of Q_i, and m_i is the positive real constant coefficients that lies in $(0, 1)$. Since P_{ij} is the intersection of L_i, L_j and every term in φ contains at least one of L_i and L_j, then φ passes all P_{ij}, where $1 \leq i < j \leq n$. According to the property of duality, we can know that $\hat{\varphi}$ is tangent to polygon $Q_1Q_2Q_3 \cdots Q_n$, and the tangent points of $\hat{\varphi}$ depend on the tangent lines of φ at P_{ij}. This means the tangent points of $\hat{\varphi}$ can be determined using m_i and $Q_i (1 \leq i \leq n)$. Let the tangent point on Q_iQ_j be Q_{ij}. To get the homogenous coordinate of the tangent point of $\hat{\varphi}$, we just need to find the homogenous coordinate of the tangent line of φ. We can write $\varphi = (m_i L_j + m_j L_i)X + L_i L_j Y$, where X, Y are products of polynomials.

$$\nabla\varphi(P_{ij}) = (m_i Q_j + m_j Q_i)X(P_{ij})$$

Then, we normalize the equation, and get $\nabla\varphi(P_{ij}) = \frac{m_i}{m_i+m_j} Q_j + \frac{m_j}{m_i+m_j} Q_i$.

Therefore, with the information of Q_i and m_i, we can get the inscribed ellipses $\hat{\varphi}$ using the Linfield's Function. (Shown in Figure 2)

In this paper we will use the Linfield's function and methods in projective geometry to extend the following theorem in [1]:

Theorem 1 *Ellipses inscribed in convex non-degenerated n-gons:*

(1) In triangles, there exists a two-parameter family of inscribed ellipses.

(2) In quadrilaterals, there exists a one-parameter family of inscribed ellipses.

(3) In pentagons, there exists a zero-parameter family of inscribed ellipse.

(4) For $n \geq 6$, if there exists inscribed ellipse, it is unique.

Also we will refine the proofs provided in [1].

2.2 In triangles, there exists a unique two-parameter family of inscribed ellipses.

Let T denote the triangle with vertices Q_1, Q_2, Q_3. Using the Linfield's function, we can get a formula for φ

$$\varphi = m_1 L_2 L_3 + m_2 L_1 L_3 + m_3 L_1 L_2$$

To set the three unknown coefficients m_1, m_2, m_3, we need to fix two parameters $0 < r, s < 1$. So that

$$\frac{m_1}{m_2} = \frac{r}{1-r}, \frac{m_2}{m_3} = \frac{s}{1-s}$$

Since we just concern about the ratio, we can set $m_2 = 1$. Then, $m_1 = \frac{r}{1-r}, m_3 = \frac{1-s}{s}$. So

$$\varphi = \frac{r}{1-r} L_2 L_3 + L_1 L_3 + \frac{1-s}{s} L_1 L_2$$

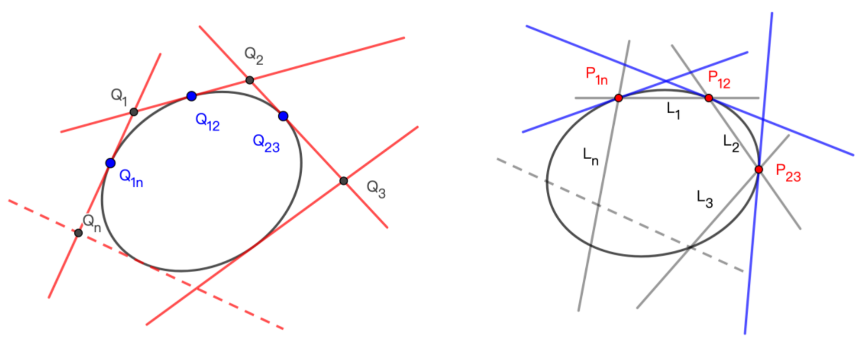

Figure 2: Explanation of Two Dual Plane

We fix r, s by fixing the tangent point on T. Let the points at which $\hat{\varphi}$ tangent to T be Q_{12} (on side Q_1Q_2) and Q_{23} (on side .Q_2Q_3). $Q_{12} = (1-r)Q_1 + rQ_2, Q_{23} = (1-s)Q_2 + sQ_3$. Since φ is quadratic, $\hat{\varphi}$ is also quadratic, and it tangent to T at all three sides. As r, s are all changeable parameters, the inscribed ellipses form a two-parameter family. (Shown in Figure 3)

Because the inscribed ellipses depend on two parameters, then we can set up the relationship between these two parameters by letting the ellipses $\hat{\varphi}$ pass a certain point. Then we will get a unique family of one-parameter ellipse that is inscribed in the triangle T. This part will show explicitly in 2.2.2.

2.2.1 An Example of Triangle Case

We can set $Q_1 = [-1 : 0 : 1], Q_2 = [1 : 0 : 1], Q_3 = [0 : 1 : 1]$, and Linfield's function is $\varphi(\alpha,\beta,\gamma) = m_3(-\alpha+\gamma)(\alpha+\gamma) + m_2(-\alpha+\gamma)(\beta+\gamma) + m_1(\alpha+\gamma)(\beta+\gamma)$. Denote (x, y, h) as a point on $\hat{\varphi}$, so $(x, y, h) = \nabla\varphi(\alpha,\beta,\gamma)$. Therefore, we can get

$$x = -2m_3\alpha + (m_1 - m_2)\beta + (m_1 - m_2)\gamma$$
$$y = (m_1 - m_2)\alpha + (m_1 + m_2)\gamma$$
$$h = (m_1 - m_2)\alpha + (m_1 + m_2)\beta + 2(m_1 + m_2 + m_3)\gamma$$

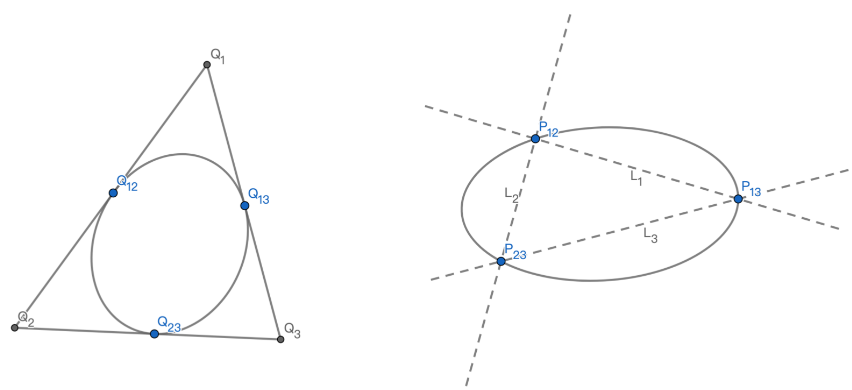

Figure 3: Triangle Case

Solving the equations above, we can get

$$\begin{aligned}
8m_1m_2m_3\alpha =& -(m_1+m_2)^2x - (m_1^2 - m_2^2 + 2m_3(m_1-m_2))y + (m_1^2 - m_2^2)h \\
8m_1m_2m_3\beta =& (m_1^2 - m_2^2 + 2m_1m_3 - 2m_2m_3)x - (m_1^2 + m_2^2 + 4m_3^2 - 2m_1m_2 + \\
& 4m_1m_3 + 4m_2m_3)y + (m_1^2 + m_2^2 - 2m_1m_2 + 2m_1m_3 + 2m_2m_3)h \\
8m_1m_2m_3\gamma =& (m_1^2 - m_2^2)x + (m_1^2 + m_2^2 - 2m_1m_2 + 2m_1m_3 + 2m_2m_3)y - \\
& (m_1^2 + m_2^2 - 2m_1m_2)h
\end{aligned}$$

Because $m_1m_2m_3 \neq 0$, the equations can be simplified to be

$$\begin{aligned}
\alpha =& -(m_1+m_2)^2x + (m_1^2 - m_2^2 + 2m_1m_3 - 2m_2m_3)y - (m_1^2 - m_2^2)h \\
\beta =& (m_1^2 - m_2^2 + 2m_1m_3 - 2m_2m_3)x - (m_1^2 + m_2^2 + 4m_3^2 - 2m_1m_2 + \\
& 4m_1m_3 + 4m_2m_3)y + (m_1^2 + m_2^2 - 2m_1m_2 + 2m_1m_3 + 2m_2m_3)h \\
\gamma =& (m_1^2 - m_2^2)x + (m_1^2 + m_2^2 - 2m_1m_2 + 2m_1m_3 + 2m_2m_3)y - (m_1^2 + m_2^2 - 2m_1m_2)h
\end{aligned}$$

Substitute these values into φ, we can get $\hat{\varphi}$

$$\begin{aligned}\hat{\varphi} =& -4m_1m_2m_3(h^2(m_1-m_2)^2+m_1^2(x+y)^2+2m_1(x+y)(m_2(x-y)+2m_3y)\\&+(2m_3y+m_2(-x+y))^2-2h(2m_1(-m_2+m_3)y+m_1^2(x+y)+m_2(-m_2x\\&+m_2y+2m_3y)))\end{aligned}$$

De-homogenize the formula we can get

$$\begin{aligned}\hat{\varphi} =& -4m_1m_2m_3((m_1-m_2)^2+m_1^2(x+y)^2+2m_1(x+y)(m_2(x-y)+2m_3y)\\&+(2m_3y+m_2(-x+y))^2-2(2m_1(-m_2+m_3)y+m_1^2(x+y)+m_2(-m_2x\\&+m_2y+2m_3y)))\end{aligned}$$

Substitute m_1 for $\frac{r}{1-r}, m_2$ for $1, m_3$ for $\frac{1-s}{s}$.

$$\begin{aligned}\hat{\varphi} =& -\frac{1}{(-1+r)^3s^3}4r(-1+s)(4(-1+r)^2y^2-4(-1+r)sy(-1+(-1+2r)x+(-1+2r)y)\\&+s^2((1+x+y)^2-4r(1+x-y+2xy+2y^2)+r^2(4+8(-1+x)y+8y^2)))\end{aligned}$$

Setting $(r,s)=(\frac{2}{7},\frac{1}{4}),(\frac{1}{2},\frac{1}{3}),(\frac{2}{3},\frac{3}{5})$, we can get the flowing picture, which we can see that the ellipse $\hat{\varphi}$ are tangent to the triangle. (Shown in Figure 4: Ellipse1: $(r,s)=(\frac{2}{7},\frac{1}{4})$, Ellipse2: $(r,s)=(\frac{1}{2},\frac{1}{3})$, Ellipse3: $(r,s)=(\frac{2}{3},\frac{3}{5})$)

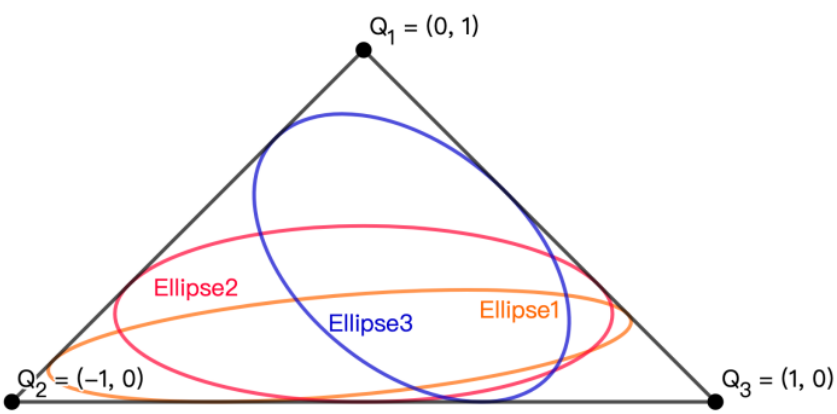

Figure 4: The Example of Triangle Case

2.2.2 The Extension of Triangle Case Example

There are still two possible families of ellipse in the case of triangle. Since the constraints we have put on the ellipse are about the tangent points, we now try to add constraints about fix points that the ellipse passes through. This will involve considering two projective planes simultaneously.

Let the ellipse $\hat{\varphi}$ in the example pass $[0 : \frac{1}{2} : 1]$. Then we can get

$$\hat{\varphi}_{[0:\frac{1}{2}:1]} = -\frac{1}{(-1+r)^3 s^3} 4r(-1+s)(4(-1+r)^2(\frac{1}{2})^2 - 4(-1+r)s(\frac{1}{2})(-1+(-1+2r)(\frac{1}{2})) + s^2((1+\frac{1}{2})^2 - 4r(1-\frac{1}{2}+2(\frac{1}{2})^2) + r^2(4-8(\frac{1}{2})+8(\frac{1}{2})^2))) = 0$$

Simplifying it, we will get

$$-\frac{1}{(-1+r)^3 s^3} r(-1+s)(4-8r+4r^2-12s+20rs-8r^2s+9s^2-16rs^2+8r^2s^2) = 0$$

From this equation, we can get a relationship between r and s.

$$s = \frac{2(2r^2-5r+3-2\sqrt{-r^4+3r^3-3r^2+r})}{8r^2-16r+9}$$

or

$$s = \frac{2(2r^2-5r+3+2\sqrt{-r^4+3r^3-3r^2+r})}{8r^2-16r+9}$$

Then we can reduce the original expression of $\hat{\varphi}$ into an expression that only relies on one parameter r.

When $s = \frac{2(2r^2-5r+3-2\sqrt{-r^4+3r^3-3r^2+r})}{8r^2-16r+9}$, plug in $r = \frac{1}{3}$, and we can get the ellipse:

$$\frac{1}{8}(-9(2\sqrt{2}+3)x^2 + 2x(2(9\sqrt{2}+13)y - 6\sqrt{2} - 9) - (2y-1)(6(12\sqrt{2}+17)y - 2\sqrt{2} - 3)) = 0$$

This is shown in Figure 5 (Ellipse 1).

Similarly, when $s = \frac{2(2r^2-5r+3+2\sqrt{-r^4+3r^3-3r^2+r})}{8r^2-16r+9}$, plug in $r = \frac{1}{3}$, and we can get the the ellipse:

$$\frac{1}{8}(9(2\sqrt{2}-3)x^2 - 2x(2(9\sqrt{2}-13)y - 6\sqrt{2} + 9) + (2y-1)(6(12\sqrt{2}-17)y - 2\sqrt{2} + 3)) = 0$$

This is shown in Figure 5 (Ellipse 2).

Actually, we can make $\hat{\varphi}$ to pass another point to determine the value of r, but we can not ensure that there is always a real solution to the equation. However, there will always be an ellipse that is tangent to three non-parallel lines and pass two distinct points in the complex plane.

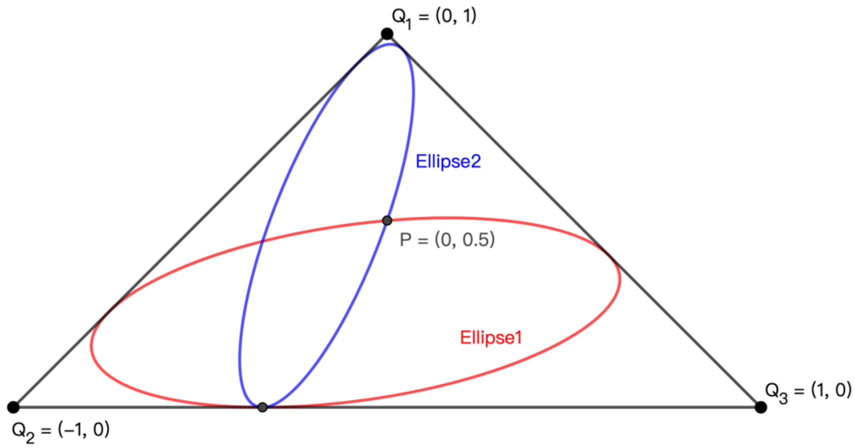

Figure 5: The Extension of Triangle Case Example

2.3 In quadrilaterals, there exists a unique oneparameter family of inscribed ellipses.

Let Q denote the quadrilateral with vertices Q_1, Q_2, Q_3, Q_4. Using the Linfield's function, we can get a formula for φ

$$\varphi = m_1 L_2 L_3 L_4 + m_2 L_1 L_3 L_4 + m_3 L_1 L_2 L_4 + m_4 L_1 L_2 L_3$$

WLOG, we can assume that the intersection of diagonals Q_2Q_4, Q_1Q_3 is the origin.(shown in Figure 6) Therefore, we will have two constraint

$$(1-\theta)Q_1 + \theta Q_3 = [0:0:1], (1-\phi)Q_2 + \phi Q_4 = [0:0:1] \tag{1}$$

where $0 < \theta, \phi < 1$.

So we just need to fix one parameter $0 < r < 0$, and

$$\frac{m_1}{m_2} = \frac{r}{1-r}, \frac{m_2}{m_4} = \frac{\phi}{1-\phi}, \frac{m_1}{m_3} = \frac{\theta}{1-\theta}$$

Let $m_2 = 1$, we can get $m_1 = \frac{r}{1-r}, m_3 = \frac{r(1-\theta)}{\theta(1-r)}, m_4 = \frac{1-\phi}{\phi}$.

We can write out the dual of the constraints in equation 1

$$(1-\theta)L_1 + \theta L_3 = \gamma, (1-\phi)L_2 + \phi L_4 = \gamma$$

Then we can represent L_2, L_3 using L_4, L_1

$$L_3 = \frac{\gamma - (1-\theta)L_1}{\theta}, L_2 = \frac{\gamma - \phi L_4}{1-\phi}$$

Therefore,

$$\varphi = (m_2L_4 + m_4L_2)L_1L_3 + (m_1L_3 + m_3L_1)L_4L_2 = \frac{\gamma}{\phi}L_1L_3 + \frac{r\gamma}{\theta(1-r)}L_4L_2$$

Then, $\varphi = \frac{\gamma}{(1-r)\theta\phi}((1-r)\theta L_1L_3 + r\phi L_2L_4)$. The dual of the first part $\frac{\gamma}{(1-r)\theta\phi}$ is the origin and the dual of the second part $(1-r)\theta L_1L_3+$ $r\phi L_2L_4$ is an ellipse that tangent to the four sides of Q from the interior.

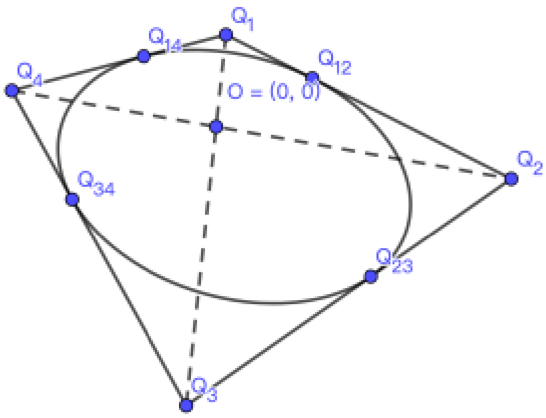

Figure 6: Quadrilateral Case

2.3.1 An Example of Quadrilateral Case

Let the original quadrilateral be $A_1A_2A_3A_4$, with $A_1 = [2:2:1], A_2 = [3:1:1], A_3 = [0:0:1], A_4 = [0:1:1]$. In order to put the intersection of the diagonals to the origin, we translate the quadrilateral into $Q_1Q_2Q_3Q_4$, with $Q_1 = [1:1:1], Q_2 = [2:0:1]$, $Q_3 = [-1:-1:1], Q_4 = [-1:0:1]$. Then $\theta = \frac{1}{2}, \phi = \frac{2}{3}$. Suppose $m_2 = 1$, we can get $m_1 = \frac{r}{1-r}, m_2 = 1, m_3 = \frac{r(1-\theta)}{\theta(1-r)}, m_4 = \frac{1-\phi}{\phi}$.

$$\varphi = (m_2L_4 + m_4L_2)L_1L_3 + (m_1L_3 + m_3L_1)L_4L_2 = \frac{3}{2}\gamma L_1L_3 + \frac{2r}{1-r}\gamma L_4L_2$$
$$= \frac{3\gamma}{1-r}(\frac{1}{2}(1-r)L_1L_3 + \frac{2}{3}rL_4L_2)$$

So the quadratic part of is

$$\varphi = \frac{1}{2}(1-r)(\alpha+\beta+\gamma)(-\alpha-\beta+\gamma) + \frac{2}{3}r(-\alpha+\gamma)(2\alpha+\gamma)$$

Then,

$$\varphi = -\frac{\alpha^2}{2} - \alpha\beta - \frac{\beta^2}{2} + \frac{\gamma^2}{2} - \frac{5\alpha^2r}{6} + \alpha\beta r + \frac{2\alpha\gamma r}{3} + \frac{\beta^2r}{2} + \frac{\gamma^2r}{6}$$

Denote (x, y, h) as a point on $\hat{\varphi}$. Because $\hat{\varphi} = \nabla\varphi$, then $(x, y, h) = \nabla\varphi(\alpha, \beta, \gamma)$. As a result, we can get

$$\begin{aligned} x &= -\frac{1}{3}\alpha(5r+3) + \beta(r-1) + \frac{2\gamma r}{3} \\ y &= (r-1)(\alpha+\beta) \\ h &= \frac{2r}{3}\alpha + \frac{3+r}{3}\gamma \end{aligned}$$

Then,

$$\begin{pmatrix} x \\ y \\ h \end{pmatrix} = \begin{pmatrix} -\frac{3+5r}{3} & ,-1+r & ,\frac{2r}{3} \\ -1+r & ,-1+r & ,0 \\ \frac{2r}{3} & ,0 & ,\frac{3+r}{3} \end{pmatrix} \cdot \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}$$

Solving this matrix, we can get

$$(-\frac{4r^3}{3} - \frac{4r^2}{3} + \frac{8r}{3})\alpha = (\frac{r^2}{3} + \frac{2r}{3} - 1)x + (-\frac{r^2}{3} - \frac{2r}{3} + 1)y + (-\frac{2r^2}{3} + \frac{2r}{3})h$$

$$(-\frac{4r^3}{3} - \frac{4r^2}{3} + \frac{8r}{3})\beta = (-\frac{r^2}{3} - \frac{2r}{3} + 1)x + (-r^2 - 2r - 1)y + (\frac{2r^2}{3} - \frac{2r}{3})h$$

$$(-\frac{4r^3}{3} - \frac{4r^2}{3} + \frac{8r}{3})\gamma = (-\frac{2r^2}{3} + \frac{2r}{3})x + (\frac{2r^2}{3} - \frac{2r}{3})y + (-\frac{8r^2}{3} + \frac{8r}{3})h$$

Because $0 < r < 1, -\frac{4r^3}{3} - \frac{4r^2}{3} + \frac{8r}{3} \neq 0$.

Simplify the equations

$$\begin{aligned} \alpha &= (r^2 + 2r - 3)x + (-r^2 - 2r + 3)y + (-2r^2 + 2r)h \\ \beta &= (-r^2 - 2r + 3)x + (-3r^2 - 6r - 3)y + (2r^2 - 2r)h \\ \gamma &= (-2r^2 + 2r)x + (2r^2 - 2r)y + (-8r^2 + 8r)h \end{aligned}$$

Substituting these value into φ, we will get the dual of φ

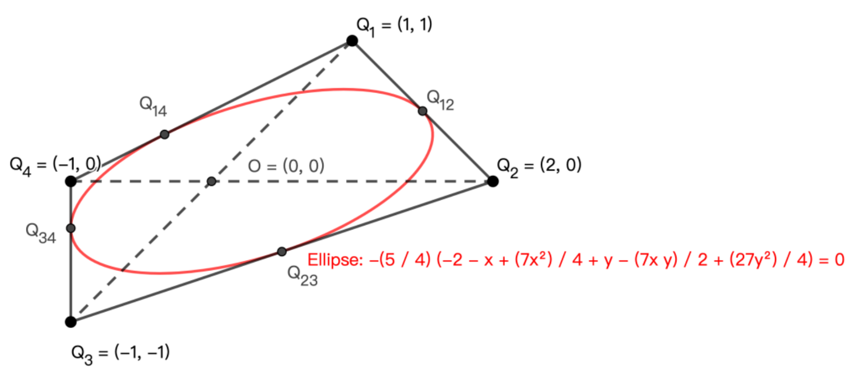

Figure 7: The Example of Quadrilateral Case

$$\hat{\varphi} = 2r(r^2+r-2)(8h^2(r-1)r + 4h(r-1)r(x-y) - (r^2+2r-3)x^2 + 2(r^2+2r-3)xy + 3(r+1)^2y^2)$$

De-homogenizing the formula, we can get

$$\hat{\varphi} = 2r(r^2+r-2)(8(r-1)r + 4(r-1)r(x-y) - (r^2+2r-3)x^2 + 2(r^2+2r-3)xy + 3(r+1)^2y^2)$$

Let $r = \frac{1}{2}$, we can get Figure 7.

2.4 In pentagons, there exists a unique zero-parameter family of inscribed ellipses.

Let P denote the pentagon with vertices Q_1, Q_2, Q_3, Q_4, Q_5. Using the Linfield's function, we can get a formula for φ

$$\varphi = m_1L_2L_3L_4L_5 + m_2L_1L_3L_4L_5 + m_3L_1L_2L_4L_5 + m_4L_1L_2L_3L_5 + m_5L_1L_2L_3L_4$$

First, for every pentagon $Q_1Q_2Q_3Q_4Q_5$, we can extend side Q_1Q_5 and Q_3Q_4 that will interact at Q_0. Then the big ellipse can be seen as inscribed in quadrilateral $Q_0Q_1Q_2Q_3$. On the other hand, we can generate $Q_1Q_2Q_3Q_4Q_5$ by adding an edge Q_4Q_5 that is tangent to the ellipse, and Q_4 is on Q_0Q_3 and Q_5 is on Q_0Q_1. Therefore, we just need to show that the ellipse we get from the quadrilateral case can be set to tangent to line Q_4Q_5. (Shown in Figure 8)

From the quadrilateral case, we know that we can write the big ellipse as $\varphi_1 = (1-r)\theta L_1L_3 + r\phi L_0L_2$. Because of the property of duality, $L_1L_3(P_{45}) = Q_4Q_5(Q_1)Q_4Q_5(Q_3)$. Since Q_1, Q_3 are on the same side of line $Q_4Q_5, Q_4Q_5(Q_1)$ and $Q_4Q_5(Q_3)$ are both positive or negative. Then $L_1L_3(P_{45}) = Q_4Q_5(Q_1)Q_4Q_5(Q_3) > 0$. Similarly, because of the property of duality, $L_0L_2(P_{45}) = Q_4Q_5(Q_0)Q_4Q_5(Q_2)$. Since Q_0, Q_2 are on the different sides of line $Q_4Q_5, Q_4Q_5(Q_0)$ and $Q_4Q_5(Q_2)$ have one positive and one negative number. Then $L_0L_2(P_{45}) = Q_4Q_5(Q_0)Q_4Q_5(Q_2) < 0$. As a result, we can always find a $r \in (0,1)$ such that $\varphi_1(P_{45}) = (1-r)\theta L_1L_3(P_{45}) + r\phi L_0L_2(P_{45}) = 0$, which means that φ_1 pass the dual of line Q_4Q_5. So $\hat{\varphi}_1$ tangent to five sides of $Q_1Q_2Q_3Q_4Q_5$.

Since φ and φ_1 are both tangent to the pentagon P, we know that

$$\nabla\varphi(P_{12}) \propto \nabla\varphi_1(P_{12}); \nabla\varphi(P_{23}) \propto \nabla\varphi_1(P_{23});$$
$$\nabla\varphi(P_{015}) \propto \nabla\varphi_1(P_{015}); \nabla\varphi(P_{034}) \propto \nabla\varphi_1(P_{034})$$

As φ is a curve of fourth power, φ_1 is proportional to φ with these constraints. Therefore, φ has a quadratic branch that is tangent to the pentagon $Q_1Q_2Q_3Q_4Q_5$.

2.4.1 An Example of Pentagon Case

Consider the pentagon $Q_1Q_2Q_3Q_4Q_5$, where $Q_1 = [0:2:1], Q_2 = [1:0:1], Q_3 = [0:-2:1], Q_4 = [-1:-1:1], Q_5 = [-1:1:1]$. Then

$$\begin{aligned}\varphi =& m_1(\alpha+\gamma)(-2\beta+\gamma)(-\alpha-\beta+\gamma)(-\alpha+\beta+\gamma)\\ &+ m_2(2\beta+\gamma)(-2\beta+\gamma)(-\alpha-\beta+\gamma)(-\alpha+\beta+\gamma)\\ &+ m_3(2\beta+\gamma)(\alpha+\gamma)(-\alpha-\beta+\gamma)(-\alpha+\beta+\gamma)\\ &+ m_4(2\beta+\gamma)(\alpha+\gamma)(-2\beta+\gamma)(-\alpha+\beta+\gamma)\\ &+ m_5(2\beta+\gamma)(\alpha+\gamma)(-2\beta+\gamma)(-\alpha-\beta+\gamma)\end{aligned}$$

Extend Q_1Q_5, Q_3Q_4 and meet at Q_0, so $Q_0 = [-2:0:1]$.

From the quadrilateral case, we can know φ_1 is inscribed in $Q_0Q_1Q_2Q_3$. In this case, $\theta = \frac{1}{2}, \phi = \frac{1}{3}$. So $\varphi_1 = \frac{1-r}{2}L_1L_3 + \frac{r}{3}L_0L_2 = \frac{1-r}{2}(2\beta+\gamma)(-2\beta+\gamma) + \frac{r}{3}(-2\alpha+\gamma)(\alpha+\gamma)$. Now, I just need to prove that φ_1 pass the dual of Q_4Q_5, which is P_{45}. Because the expression for Q_4Q_5 is $\alpha+\gamma = 0$, then $P_{45} = [1:0:1]$. Therefore $\varphi_1(P_{45}) = \frac{1-r}{2}(1)(1) + \frac{r}{3}(-1)(2) = \frac{1-r}{2} - \frac{2}{3}r$. As a result, $r = \frac{3}{7}$ satisfy the requirement that φ_1 pass P_{45}.

$$\varphi_1 = \frac{1}{7}(-2\alpha^2 - 8\beta^2 - \alpha\gamma + 3\gamma^2)$$

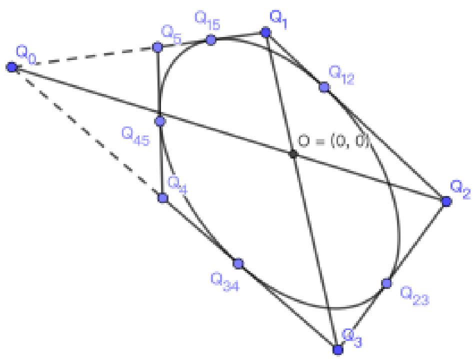

Figure 8: Pentagon Case

Denote (x, y, h) as a point on $\hat{\varphi}_1$. Because $\hat{\varphi}_1 = \nabla\varphi_1$, then $(x, y, h) = \nabla\varphi_1(\alpha, \beta, \gamma)$. As a result, we can get

$$x = -\frac{4}{7}\alpha - \frac{1}{7}\gamma$$
$$y = -\frac{16}{7}\beta$$
$$h = -\frac{1}{7}\alpha + \frac{6}{7}\gamma$$

Solve these equations for α, β, γ :

$$\frac{400}{343}\alpha = -\frac{96}{49}x - \frac{16}{49}h$$
$$\frac{400}{343}\beta = -\frac{25}{49}y$$
$$\frac{400}{343}\gamma = -\frac{16}{49}x + \frac{64}{49}h$$

Since we just consider the ratio, we can simplify the equations into

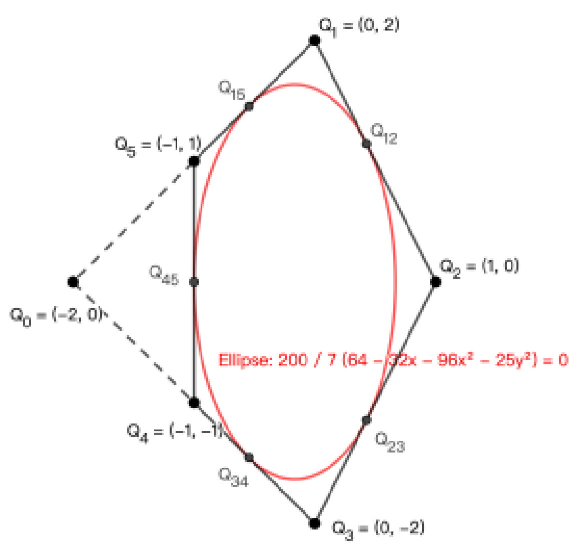

Figure 9: The Example of Pentagon Case

Therefore,

$$\hat{\varphi}_1 = -\frac{200}{7}(96x^2 + 25y^2 + 32xh - 64h^2)$$

De-homogenizing the formula, we can get

$$\hat{\varphi}_1 = -\frac{200}{7}(96x^2 + 25y^2 + 32x - 64)$$

It is the inscribed ellipse we want. (Figure 9)

2.4.2 Future Direction

Actually, we can deduce quadrilateral case based on triangle case using the similar method that we use when deducing pentagon case. However, in this case we will have a completely different form of φ_1 and we can not continue the deduction to pentagon. This is a left question to answer.

The deduction from triangle to quadrilateral is produced as follow. Let Q denote the quadrilateral with vertices Q_1, Q_2, Q_3, Q_4. (Shown in Figure 10) Using the Linfield's function, we can get a formula for φ

$$\varphi = m_1 L_2 L_3 L_4 + m_2 L_1 L_3 L_4 + m_3 L_1 L_2 L_4 + m_4 L_1 L_2 L_3$$

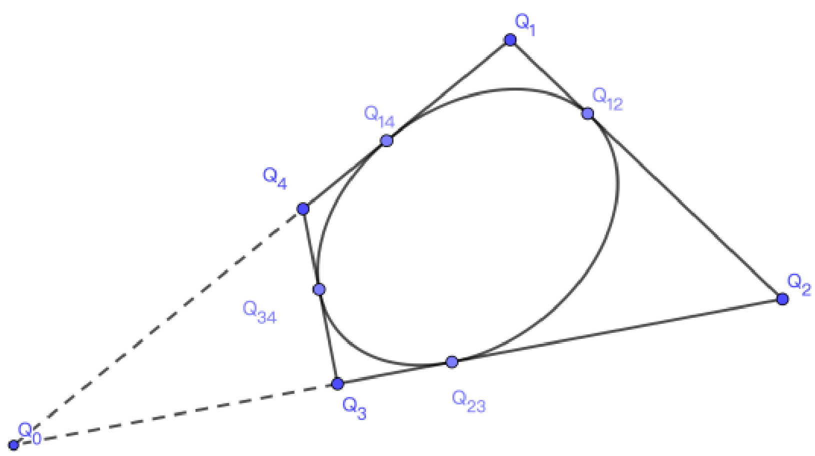

Figure 10: Explanation of The Deduction Quadrilateral Case

We can extend Q_1Q_4 and Q_2Q_3 to meet at Q_0.

From the triangular case, there is a two-parameter family of inscribed ellipse $\varphi_1 = \frac{r}{1-r} L_0 L_2 + L_0 L_1 + \frac{1-s}{s} L_1 L_2$. I will prove that we can choose proper s to make φ_1 pass the dual of Q_3Q_4, which is P_{34}.

Since Q_1, Q_2 are on the same side of Q_3Q_4 and are on the different side with Q_0, we can get $Q_3Q_4(Q_1)Q_3Q_4(Q_2) > 0, Q_3Q_4(Q_0)Q_3Q_4(Q_1) < 0$, and $Q_3Q_4(Q_0)Q_3Q_4(Q_2) < 0$, which means $L_1L_2(P_{34}) > 0, L_0L_1(P_{34}) < 0$, and $L_0L_2(P_{34}) < 0$. Because $0 < r, s < 1$, then we can get all positive real number by choosing proper r and s for $\frac{r}{1-r}, \frac{1-s}{s}$. Therefore, for every r, we can always find a corresponding s such that $\frac{r}{1-r} L_0 L_2 + L_0 L_1 + \frac{1-s}{s} L_1 L_2 = 0$

According to the formula of the tangent point, we can know that

$$\nabla\varphi(P_{12}) \propto \nabla\varphi_1(P_{12}), \nabla\varphi(P_{23}) \propto \nabla\varphi_1(P_{23}), \nabla\varphi_1(P_{14}) \propto \nabla\varphi_1(P_{14})$$

Because φ is a cubic curve, φ is proportional to φ_1

Therefore, there is a one-parameter ellipse that tangent to the four sides of Q from the interior.

2.5 Situation for N-gons (N $\geq$ 6)

If a N-gon has an inscribed ellipse $\hat{\varphi}$, then the dual conic curve φ must pass at least six points. However, since the general representation of a conic curve is $Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$, every five points can assign a unique ellipse. Let the six points that φ should pass be: $(x_1, y_1), (x_2, y_2), (x_3, y_3), (x_4, y_4), (x_5, y_5), (x_6, y_6)$. Then we can get a system of equations:

$$\begin{cases} Ax_1^2 + Bx_1y_1 + Cy_1^2 + Dx_1 + Ey_1 + F = 0 \\ Ax_2^2 + Bx_2y_2 + Cy_2^2 + Dx_2 + Ey_2 + F = 0 \\ Ax_3^2 + Bx_3y_3 + Cy_3^2 + Dx_3 + Ey_3 + F = 0 \\ Ax_4^2 + Bx_4y_4 + Cy_4^2 + Dx_4 + Ey_4 + F = 0 \\ Ax_5^2 + Bx_5y_5 + Cy_5^2 + Dx_5 + Ey_5 + F = 0 \\ Ax_6^2 + Bx_6y_6 + Cy_6^2 + Dx_6 + Ey_6 + F = 0 \end{cases}$$

To make sure this equation system has none zero solution, we need to ensure the determinant of the coefficient matrix is zero:

$$\begin{vmatrix} x_1^2 & x_1y_1 & y_1^2 & x_1 & y_1 & 1 \\ x_2^2 & x_2y_2 & y_2^2 & x_2 & y_2 & 1 \\ x_3^2 & x_3y_3 & y_3^2 & x_3 & y_3 & 1 \\ x_4^2 & x_4y_4 & y_4^2 & x_4 & y_4 & 1 \\ x_5^2 & x_5y_5 & y_5^2 & x_5 & y_5 & 1 \\ x_6^2 & x_6y_6 & y_6^2 & x_6 & y_6 & 1 \end{vmatrix} = 0$$

It is clear that we can get many coefficient matrices by choosing sets of six dual points that φ needs to pass. Thus, the sufficient and necessary condition for a n-gon to have inscribed ellipse is that every determinant of coefficient matrices is zero.

2.6 Program Realization

We used python to generate the inscribed ellipses of a given polygon, matplotlib and numpy packages were used to generate the plot.

In each case, the polygon is held fixed. We used meshgrid in the xy plane and made the contour plot for the ellipse. The contour plot level was fixed to ensure that there is a unique ellipse in the xy plane.

We designed a function to allow the user input the x-coordinate of points on the polygon, and generate the inscribed ellipse corresponding to the input. We also made a bash script that takes multiple inputs and generates multiple ellipses that correspond to certain input from the user.

The result shows that the methods provided by this paper are valid and can be used by programs to solve the problem of drawing inscribed ellipses with an assigned polygon and assigned tangent points.

3 Conclusion

In this paper, we extend the result of previous literature in three cases and use python to build a program that enables users to draw inscribed ellipses by assigning tangent points on the polygons'edges. In the triangle case, a method to limit the inscribed ellipses from two planes simultaneously is proposed. For the quadrilateral case, we simplify the conditions required for the proof through removing a rotation matrix. We also propose a new way to derive quadrilateral case from triangle case. We improve the proof using duality in the pentagon case. Examples are calculated detailedly for all three cases. Furthermore, we explore the condition for $\mathrm{n-gons}(n \geq 6)$ and use python to realize the methods provided in this paper.

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