On Higher Dimensional Orchard Visibility Problem

Shengning Zhang

Abstract. In this article, we study Pólya's orchard visibility problem in arbitrary dimension d: suppose at every integral point in $\mathbb{R}^d$, centered a small d-dimensional ball with radius r (which is considered as a tree at the integral point), given a d-dimensional ball centered at the origin O with radius R (which is considered as the orchard), it asks for the smallest r such that every ray starting from O will hit some tree in the orchard. We give both the upper and the lower bounds of the minimal value of r, say ρ in terms of R. Moreover, we prove that as $R \to \infty$, $\rho = \mathcal{O}(R^{-\frac{1}{d-1}})$.

1. Introduction

Let Λ be the set of lattice points $\mathbb{Z}^d \backslash O$ in $\mathbb{R}^d$, where O is the origin. Let $B(O,R)$ be the closed ball in $\mathbb{R}^d$ centered at O with radius $R > 1$. Centering at every integral point $P \in B(O,R)$, is a small closed ball $B(P,r)$ with given small radius $r > 0$. The original Pólya's orchard visibility problem considers the case $d = 2$, when the disc $B(O,R)$ is thought as a round orchard and every $B(P,r)$ a tree at P, it asks for the smallest r, which we denote by ρ, so that one standing at the center O cannot see through the orchard, that is, for any ray l starting from O, $l \cap B(P,r) \neq \emptyset$ for some P.

In [1], it proved that

$$\frac{1}{\sqrt{R^2+1}} < \rho < \frac{1}{R}. \tag{1.1}$$

Indeed, in an earlier paper [2], Thomas Tracy Allen had proved that

$$\rho = \frac{1}{R}. \tag{1.2}$$

In this paper, we'd like to study the general Pólya's orchard problem in arbitrary dimension d and prove similar bounds as in (1.1). Our strategy follows [3], where, however, only deals with the 2 and 3 dimensional cases.

2. Lower bounds

Consider in $\mathbb{R}^d$ the d-dimensional cuboid C with diagonal vertices O and $D := (1, 1, \cdots, 1, [R] + 1)$, where $[R]$ is the floor function of R. Then

$$C \cap \mathbb{Z}^d = \{(x_1, \cdots, x_d) \in \mathbb{Z}^d \mid x_i \in [0, 1], \forall i = 1, \cdots, d-1; x_d \in [0, [R] + 1]\}.$$

Apparently, D is not in $B(O, R)$. The segment OD is of the length $\sqrt{(d-1) + R^2}$, and any $P \in C \cap \mathbb{Z}^d$ has the distance squared $dist(P, OD)^2$ to OD

$$\frac{(d - 1 + ([R] + 1)^2)(x_1^2 + \cdots + x_d^2) - (x_1 + \cdots + x_{d-1} + ([R] + 1)x_d)^2}{d - 1 + ([R] + 1)^2} \tag{2.1}$$

This lead us to our first result, which is a direct generalization to the first inequality of (1.1).

PROPOSITION 1. *notations as above*

$$\frac{\sqrt{d-1}}{\sqrt{d - 1 + ([R] + 1)^2}} < \rho. \tag{2.2}$$

PROOF: Consider the formula (2.1), apparently that among all integral points in C other than O and D, $P_0 = \{0, \cdots, 0, 1\}$ minimize the expression, when

$$dist(P_0, OD)^2 = \frac{d-1}{d - 1 + ([R] + 1)^2}.$$

(see the figure below)

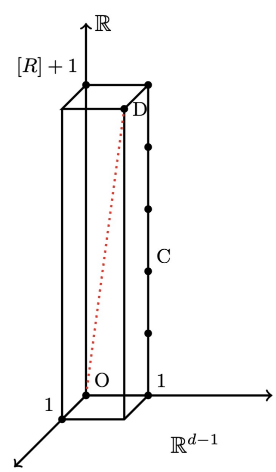

FIGURE 1

So if the tree radius r can block the orchard, it must bigger than $\frac{\sqrt{d-1}}{\sqrt{d-1+([R]+1)^2}}$. This completes the proof. □

The proposition tells us that ρ grows faster than the rate of R^{-1} as R goes to infinity, however, it is not the exact rate of growth of ρ, so we want a better lower bound of ρ in terms of R. Indeed, the proof of the proposition tells us that, to obtain such a lower bound, we have to consider a "finer" solid containing the ray than the coboid C above. To use such a solid in higher dimension, we have to use the volume formula of a lattice polyhedron in higher dimension developed by Macdonald in [4], which is the generalization of Pick's Theorem used in [3, Theorem 2.2]. Now we summarize below.

Let $\mathbb{Z}^d \subset \mathbb{R}^d$ be the standard integral lattice, X a d-dimensional polyhedra in $\mathbb{R}^d$ whose vertices are all in $\mathbb{Z}^d$. Let ∂X be the boundary of X, which can be viewed as a $(d-1)$- simplicial complex. For any integer $n > 0$, write

$$L(n, X) = |X \cap \frac{1}{n}\mathbb{Z}^d|,$$

and

$$M(n, X) = L(n, X) - \frac{1}{2}L(n, \partial X),$$

then, we have the volume of X can be computed by:

PROPOSITION 2 (Macdonald's Theorem). *The volume of the polyhedra $Vol(X)$ equals*

$$\frac{2}{(d-1)d\,!}\sum_{i=0}^{d-1}(-1)^i M(d-1-i, X)\binom{d-1}{i},$$

where $M(0, X) = 1$ if d is even, $M(0, X) = 0$ if d is odd.

Now we give us first theorem

THEOREM 1. *There is a constant $c > 0$ such that*

$$([R] + 1)\rho^{d-1} > c. \tag{2.3}$$

Remark 1. The constant c is given by the volume of a polyhedra, which can be computed using Macdonald's Theorem above. The key is to construct a proper polyhedra, which will be clear in the proof of the theorem.

LEMMA 1. *Point $Q \in \mathbb{Z}^d \cap B(O, R)$, if for any $P \in \mathbb{Z}^d \cap B(O, R)$, $OB \cap B(P, r) = \emptyset$, then the coordinates of Q are coprime, that is, if $Q = (a_1, \cdots, a_d)$ then $gcd(a_1, \cdots, a_d) = 1$.*

The lemma comes from an easy observation. Suppose $gcd(a_1, \cdots, a_d) = d > 1$, then $P_1 = \frac{1}{d}(a_1, \cdots, a_d) \in \mathbb{Z}^d \cap B(O, R)$ and obviously $OB \cap B(P_1, r) \neq \emptyset$. $\square$

LEMMA 2. *Let l be any ray starting from O, if point $P \in \mathbb{Z}^d \cap B(O, R)$, $P \notin l$ such that $dist(P, l)$ is minimal, then the coordinates of P are coprime.*

Suppose the coordinates of P are coprime with greatest common divisor $d > 1$, then $dist(\frac{1}{d}P, l) < dist(P, l)$. Contradiction. □

To carry out our argument in high dimension, we have to generalize the result to Lemma 2 from a ray l to a family of geometric objects which we called diamonds with a diagonal, and is defined as follow:

DEFINITION 1. In $\mathbb{R}^d$, for any positive integer $n \leq d$, a n-dimensional diamond $\mathfrak{D}$ with a diagonal I is defined as follow:

(1) A 1-dimensional diamond $\mathfrak{D}$ is nothing but a segment that start from the origin O to a point $P \neq O$ in $\mathbb{R}^d$ and its diagonal I is itself;
(2) Suppose for any $i \leq n$, the i-dimensional diamonds with a diagonal are well-defined, then a n-dimensional diamond $\mathfrak{D}_n$ with a diagonal I_n is defined base on some n-dimensional diamond $\mathfrak{D}_{n-1}$ with a diagonal I_{n-1}: let V_{n-1} be the $(n-1)$-dimensional vector space generated by vectors in $\mathfrak{D}_{n-1}$, and P_n a point in $\mathbb{R}^d \backslash V_{n-1}$. Consider vectors $\boldsymbol{OI_{n-1}}$ and $\boldsymbol{OP_n}$, then define Q_n be the end point of $\boldsymbol{OI_{n-1} - OP_n}$. $\mathfrak{D}_n$ is defined to be the convex hull of $\mathfrak{D}_{n-1} \cup \{P_n, Q_n\}$, and its diagonal is $I_n := I_{n-1}$.

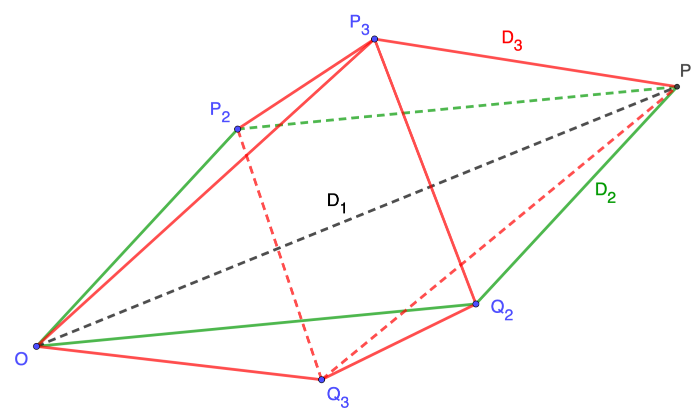

FIGURE 2. an example of 1,2 and 3-diamonds

LEMMA 3. *Let $\mathfrak{D}$ be a n-dimensional diamond with a diagonal I in $\mathbb{R}^d$, $n < d$, V be n-dimensional subspace in $\mathbb{R}^d$ generated by $\mathfrak{D}$. Now if a point $P \in \mathbb{Z}^d \cap B(O, R)$, $P \notin V$ such that $dist(P, \mathfrak{D})$ is minimal, then the coordinates of P are coprime.*

Suppose $A \in \mathfrak{D}$ is the point such that $dist(P, \mathfrak{D}) = dist(P, A) = a$. Consider the triangle ΔOAP, since $\mathfrak{D}$ is a convex hull by the definition, the segment $OA \subset \mathfrak{D}$. Now if the greatest common divisor of the coordinates of P is $m > 1$, consider the point $Q = \frac{1}{m}P \in OP$. Find a point $Q' \in OA \subset \mathfrak{D}$ such that $QQ' \parallel AP$, then apparently that $dist(Q, \mathfrak{D}) < dist(P, \mathfrak{D})$. Contradiction! $\square$

PROOF OF THE THEOREM: Consider the point $D_1 := D$ given above, we view the segment OD as a vector from the origin O to D and denote it by $\boldsymbol{l}$. Among all integral points in $B(O, R)$, find P_2 in the first quadrant (that is, all the points are of nonnegative coordinates) be the one of minimal distance to $\boldsymbol{l}$. Write the minimal distance ε_1. From the lemmas above, we know the coordinates of P_2 are coprime. View the segment OP_2 as a vector and denote it by $\boldsymbol{v}_1$, and define vector $\boldsymbol{u}_1 := \boldsymbol{l} - \boldsymbol{v}_1$, define the two dimensional diamond D_2 be the parallelogram spanned by $\boldsymbol{v}_1$ and $\boldsymbol{u}_1$. From the two lemmas above, D_2 does not contain any integral points of Λ other than the 4 vertices. Denote the 2-dimensional plane spanned by $\boldsymbol{v}_1$ and $\boldsymbol{u}_1$ by V_2. Using our notion of diamond, D_2 is a 2-dimensional diamond with a diagonal l.

Now among all integral points in $B(O, R)\backslash V_2$, find one P_3 in the first quadrant of the minimal distance to $V_2 \cap B(O, R)$. Write the minimal distance ε_2. Consider the 2-dimensional diamond D_2 with diagonal $\boldsymbol{l}$ and the point P_3, by Definition 1, they together define a 3-dimensional diamond D_3 with diagonal $\boldsymbol{l}$. By Lemma 3, all the coordinates of P_3 are coprime, D_3 contains no integral points other than the 6 vertices. Denote the 3-dimensional vector space generated by vectors in D_3 by V_3.

Keep this process, for all integer $i = 1, 2, \cdots, d$, we obtain i-dimensional diamond D_i with diagonal $\boldsymbol{l}$, $V_i = spanD_i$, integral points P_i in the first quadrant such that

(a) $dist(P_i, V_{i-1} \cap V_i) = \varepsilon_{i-1}$ is minimal among all integral points in $B(O, R)\backslash V_{i-1}$;
(b) D_i is the diamond constructed by D_{i-1} and P_i;
(c) D_i contains no integral points other than its vertices.

It is easy to see, from our construction, the volume of D_i is

$$Vol(D_i) = \frac{2^{i-1}}{i!}\varepsilon_1 \cdots \varepsilon_{i-1}([R] + 1). \tag{2.4}$$

In particular, Write $\mathfrak{D} := D - d$, its volume is

$$Vol(\mathfrak{D}) = \frac{2^{d-1}}{d!}\varepsilon_1 \cdots \varepsilon_{d-1}([R] + 1), \tag{2.5}$$

which can also be calculated by Macdonald's formula as in Proposition 2. On the other hand, By our construction of $\mathfrak{D}$, if the tree radius r is such that every ray starting from O and passing through one point in $\mathfrak{D}$ will be blocked by some tree, then $r > \varepsilon_i$ for any i. So we have

$$\frac{2^{d-1}}{d!}r^{d-1}([R] + 1) > Vol(\mathfrak{D}). \tag{2.6}$$

Writing

$$c = \frac{d! Vol(\mathfrak{D})}{2^{d-1}}, \tag{2.7}$$

we complete the proof. □

Remark 2. If $d = 2$, $Vol(\mathfrak{D}) = Vol(D_2) = 1$, then the Theorem tells that

$$([R] + 1)\rho > 1, \tag{2.8}$$

which reproduces the result in [3, Proposition 2.4].

If $d = 3$, $Vol(\mathfrak{D}) = Vol(D_3) = \frac{2}{3} \times \frac{\sqrt{2}}{2} \times \frac{\sqrt{2}}{2} = \frac{1}{3}$, then the theorem tells that

$$([R] + 1)\rho^2 > \frac{1}{2}, \tag{2.9}$$

which is better than the result in [3, Proposition 4.4].

3. Upper bounds

In this section we give an upper bound of ρ in terms of R. The key ingredient is again Minkowski's theorem as [3, Theorem 4.1], which we summarize below.

Proposition 3 (Minkowski's Theorem). *Let m be a positive integer and $F \subset \mathbb{R}^d$ a domain satisfying*

(*a*) *F is symmetric with respect to O;*
(*b*) *F is convex;*
(*c*) *$Vol(F) > m2^d$.*

Then F contains at least m pairs of points $\pm A_i \in \mathbb{Z}^d \backslash O$, $1 \le i \le m$, which are distinct from each other.

Now we state an upper bound of ρ. The idea is essentially same to [3, §4], where, however, only deals with the 3-dimensional case.

Theorem 2. *There is a constant $C > 0$, such that*

$$R\rho^{d-1} < C. \tag{3.1}$$

Proof: For any diameter AA' of the ball $B(O, R)$, let's consider the $d-1$-dimensional hyperellipsoid $E \subset \mathbb{R}^d$ as follow:

(*i*) AA' is a long axis of E;
(*ii*) all other semi-axes of E are equal of length h.

Indeed, consider the function of d variables:

$$F(x_1, \cdots, x_d) := \frac{x_1^2}{h^2} + \cdots + \frac{x_{d-1}^2}{h^2} + \frac{x_d^2}{R^2},$$

then $F(x_1, \cdots, x_d) = 1$ gives the hyperellipsoid when AA' is lying in the x_d-axis. Generally, if the line AA' has a unit directional vector $\vec{u}_d$, extend it to a orthnormal basis $\beta := \{\boldsymbol{u}_1, \cdots, \boldsymbol{u}_{d-1}, \boldsymbol{u}_d\}$ of $\mathbb{R}^d$. Then there exists a unitary transformation $T : \mathbb{R}^d \to \mathbb{R}^d$ which sends β to the standard orthnormal basis $\{(1, 0, \cdots, 0), \cdots, (0, \cdots, 0, 1)\}$ such that $T(\boldsymbol{u}_d) = (0, \cdots, 0, 1)$. Then the $(d-1)$-dimensional hyperellipsoid E has equation $F(T(x_1, \cdots, x_d)) = 1$. See Figure 3.

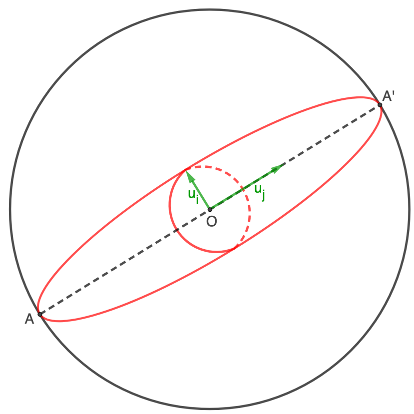

FIGURE 3

Now let $F \subset \mathbb{R}^d$ be the domain enclosed by E (including the points of E). Apparently, F satisfies the condition (a) and (b) of Minkowski's Theorem. Moreover, it is known that the volume of the hyperellipsoid is

$$\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} h^{d-1} R, \tag{3.2}$$

here Γ is the gamma function, so

$$\Gamma(\frac{d}{2}+1) = \frac{d}{2}\Gamma(\frac{d}{2}) = \frac{d}{2}(\frac{d}{2}-1)\cdots\gamma_0, \tag{3.3}$$

where $\gamma_0 = 1$ if d is even, $\gamma_0 = \frac{\pi}{2}$ if d is odd. By Minkowski's Theorem, if we choose h such that

$$\frac{\pi^{\frac{d}{2}}}{\Gamma(\frac{d}{2}+1)} h^{d-1} R = 2^d + \varepsilon, \tag{3.4}$$

where ε is an arbitrary positive real number, then F contains an integral point other than O. This implies that, if we set $C = \frac{(2^d+\varepsilon)\Gamma(\frac{d}{2}+1)}{\pi^{\frac{d}{2}}}$, and the tree radius $r = CR^{-\frac{1}{d-1}}$, then any ray segment OA starting from O will be blocked by some tree at the integral point contained in F we constructed as above. Since $\rho < r$, that we complete the proof. $\square$

Combining Theorem 1 and Theorem 2, we obtain the main result of this article:

THEOREM 3. *For d-dimensional orchard visibility problem, as the radius of orchard R goes to infinity,*

$$\rho = \mathcal{O}(R^{-\frac{1}{d-1}}). \tag{3.5}$$

4. Some Further Thoughts

We can still ask a lot of questions concerning the orchard visibility problem in arbitrary dimension. For example, for $d = 2$, it has been proved in [2] that $\rho = \frac{1}{R}$, or

$$\lim_{R\to\infty} \rho R = 1. \tag{4.1}$$

Inspired by our results, it is natural to ask if we can find a constant l for dimension d such that

$$\lim_{R\to\infty} \rho^{d-1} R = l. \tag{4.2}$$

However, our estimation in this article using polyhedra is apparently not precise and fine enough for such a conclusion. We will explore this problem in the future.

References

[1] Clyde P. Kruskal *The orchard visibility problem and some variants*, Journal of computer and system sciences, 74 (2008) 587–597.

[2] Thomas Tracy Allen *Polya's Orchard Problem*,The American Mathematical Monthly, Vol. 93, No. 2 (1986) 98-104.

[3] Alexandru Hening and Michael Kelly *On Polya's Orchard Problem*,Rose-Hulman Undergraduate Mathematics Journal: Vol. 7 : Iss. 2 , Article 9.

[4] I. G. Macdonald *The volume of a lattice polyhedron*, Mathematical Proceedings of the Cambridge Philosophical Society, 59 (1963) 719-726.